PUPT-1690
UPR-744-T
ILL-(TH)-97-3
hep-th/9704143

Branes at Angles and Black Holes

Vijay Balasubramanian *
Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138

Finn Larsen †
David Rittenhouse Laboratories
University of Pennsylvania
Philadelphia, PA 19104

Robert G. Leigh ‡
Department of Physics
University of Illinois at Urbana-Champaign
Urbana, IL 61801.

Abstract

We construct the most general supersymmetric configuration of D2-branes and D6-branes on a 6-torus. It contains arbitrary numbers of branes at relative $U(3)$ angles. The corresponding supergravity solutions are constructed and expressed in a remarkably simple form, using the complex geometry of the compact space. The spacetime supersymmetry of the configuration is verified explicitly, by solution of the Killing spinor equations, and the equations of motion are verified too. Our configurations can be interpreted as a 16-parameter family of regular extremal black holes in four dimensions. Their entropy is interpreted microscopically by counting the degeneracy of bound states of D-branes. Our result agrees in detail with the prediction for the degeneracy of BPS states in terms of the quartic invariant of the E(7,7) duality group.

*vijayb@pauli.harvard.edu
†larsen@cvetic.hep.upenn.edu
‡rgleigh@uiuc.edu

1 Introduction

In recent work the description of the p-brane solitons of supergravity theory at weak coupling as D-branes in string theory has been exploited to elucidate many nonperturbative aspects of string theory. An important strategy has been the construction of complicated solutions to the supergravity equations of motion by combination of simpler building blocks that can be unambiguously identified with individual branes. (See [1] for example.) Such composite structures are naturally identified with bound states of the elementary constituents. A striking success of this reasoning has been the representation of a large class of extremal black holes as bound states of p-branes. This has led to the interpretation of the entropy of these black holes in terms of the bound state degeneracy of the corresponding system of branes [2, 3, 4, 5, 6].

However, the early study of bound states of p-branes was limited because it was only possible to construct supersymmetric configurations corresponding to branes that were *orthogonal* to each other. It is now known that in fact there are also supersymmetric bound states where the component branes are at relative *angles* [7, 8]. Very recently, explicit classical solutions to the supergravity equations that correspond to some of these configurations of branes at angles have been presented in the literature [9, 10, 11, 12, 13].

In this paper, we construct configurations of branes which cover extended regions of the parameter space of branes at angles. We begin, in Sec. 2, by considering Type IIA string theory compactified on the 6-torus T^6, and construct a new class of supersymmetric bound states of D2-branes oriented at arbitrary $U(3)$ angles in the presence of an arbitrary number of D6-branes. Previous constructions involving D2-brane at angles are a subset of those presented here [7, 8]. Indeed, we find the most general class of supersymmetric bound states of D2-branes and D6-branes permissible on T^6.

In Sec. 3 we present the corresponding classical solutions. They have a remarkably simple structure that becomes apparent when they are displayed in terms of the complex geometry of the compact space: branes at angles are simply described by the superposition of two-forms aligned with the constituent branes, with the superposition coefficients taken to be harmonic functions on the noncompact space. The two-form resulting from the superposition characterizes the ensemble of branes and enters the internal metric as a simple modification of its Kähler form. This result generalizes the "harmonic function rule" for construction of orthogonally intersecting branes to a setting involving branes at angles [14]. It is remarkable that the general case presented here may be cast in such a simple form, simpler than the special cases known hitherto. In Sec. 4 it is verified explicitly that our configurations indeed solve the Killing spinor equations of supergravity. This requires a strategy that takes advantage of both the diagonal spacetime metric and the complex geometry of the compact space. The explicit verification of the equations of motion, presented in Appendix B, similarly uses the complex geometry heavily. It is apparent that our construction applies to black holes on general Calabi-Yau manifolds, and we will develop the details in a forthcoming article [15].

Generically our classical solutions can be interpreted as regular four-dimensional

black holes with finite horizon area. Indeed this is the main motivation for our investigation. The most general black hole that couples to electric RR 3-form and RR 7-form gauge fields is parametrized by 16 charges: the projections of the 2-form charges on the 15 2-cycles of the 6-torus, and the 6-form charge. The black holes that we construct account for the complete 16 dimensional phase space. T-dualizing the entire T^6 converts the D2-branes and D6-branes into D4-branes and D0-branes allowing us to analyze another complete 16 parameter subspace of the extremal black hole phase space. The most general Type IIA black hole carrying purely Ramond-Ramond charges can be understood as an amalgam of these two constructions. Their combination involves fluxes on the D-brane world-volumes. We leave the general analysis of classical solutions corresponding to D-branes with fluxes to a future paper [15] and conclude this work in Sec. 5 by showing that the entropy of our general black holes can be accounted for microscopically in terms of the corresponding D-brane bound state degeneracy.

2 D-branes at angles

In this section we use the techniques of [7, 8] to construct supersymmetric configurations of D2-branes and D6-branes on a 6-torus. We find that a general supersymmetric state of D2-branes on T^6 can be constructed by placing an arbitrary number of branes at relative $U(3)$ angles[1]. Then we show that a D6-brane can always be added to such a configuration without breaking supersymmetry. T-dualizing the entire torus yields the general supersymmetric state of D4-branes and D0-branes on T^6. Finally, an explicit example with three D2-branes and a D6-brane is developed in detail, and special cases are compared with results from the literature.

2.1 Preliminaries

In this section we work in light-cone frame where the two supersymmetries of Type II string theory, Q and $\tilde{Q}$, are 16-component chiral $SO(8)$ spinors. A Dp-brane imposes the following projection on Q and $\tilde{Q}$ [16, 7, 8] [2]:

$$Q \pm \Omega_p(\gamma)\tilde{Q} = 0 \tag{1}$$

Here Ω is the volume form of the brane:

$$\Omega_p(\gamma) = \frac{1}{(p+1)!}\epsilon_{i_0\cdots i_p}\gamma^{i_0}\cdots\gamma^{i_p} \tag{2}$$

Note that this operator is normalized to be a projection operator. The $\pm$ signs in Eq. 1 distinguish between branes and anti-branes or, equivalently, between opposite

[1] The results in this section are also valid before compactification. In general the branes of such configurations are localized in the internal dimensions. We refer to the compactified case to ease the comparison with results presented in later sections.

[2] We take $\Gamma_{11}\tilde{Q} = +\tilde{Q}$.

orientations of the Dp-brane. For simplicity, we choose the moduli of the torus to be unity, *i.e.* the torus is square[3].

We are interested in configurations composed of many D-branes. Here supersymmetry demands that the projections imposed by each brane are simultaneously satisfied. To solve the resulting system of equations it is convenient to introduce complex coordinates (z_1, z_2, z_3) on the 6-torus which are related to the real coordinates as $z_\mu = (y_{2\mu-1} + iy_{2\mu})/\sqrt{2}$, $\mu = 1, 2, 3$. The corresponding complexified Gamma matrices are $\Gamma^\mu = (\gamma^{2\mu-1} + i\gamma^{2\mu})/2$ and their complex conjugates $\bar{\Gamma}^{\bar{\mu}} = (\gamma^{2\mu-1} - i\gamma^{2\mu})/2$. These matrices obey a Clifford algebra[4]:

$$\{\Gamma^\mu, \Gamma^\nu\} = \{\bar{\Gamma}^{\bar{\mu}}, \bar{\Gamma}^{\bar{\nu}}\} = 0 \quad ; \qquad \{\Gamma^\mu, \bar{\Gamma}^{\bar{\nu}}\} = \delta^{\mu\bar{\nu}} \tag{3}$$

Taking z_4 and $\bar{z}_4$ to be complex coordinates describing the remaining two transverse directions in the light-cone frame we also define the corresponding complexified Gamma matrices Γ^4 and $\bar{\Gamma}^{\bar{4}}$. These Γ matrices also obey a Clifford algebra with the Γ^μ in Eq. 3.

The 16-component SO(8) chiral spinors $\tilde{Q}$ can be represented in terms of a Fock basis $|n_1, n_2, n_3\rangle \otimes |n_4\rangle$ on which the Γ_μ ($\bar{\Gamma}_{\bar{\mu}}$) and Γ_4 ($\bar{\Gamma}_{\bar{4}}$) act as annihilation (creation) operators. Specifically:

$$\bar{\Gamma}_{\bar{\mu}} \Gamma_\mu |n_1, n_2, n_3\rangle \otimes |n_4\rangle = n_\mu |n_1, n_2, n_3\rangle \otimes |n_4\rangle \tag{4}$$

where the n_μ and n_4 take values 0 and 1. In this basis it will be simple to find spinors that satisfy the projections imposed by the branes. It will be useful to know how $\tilde{Q}$ transforms under the $SO(6) \subset SO(8)$ rotations acting on the 6-torus on which the branes are wrapped. Since $|n_4\rangle$ is inert under these rotations, the 16-component $SO(8)$ spinor $\tilde{Q}$ transforms in the 8-dimensional spinor representation of $SO(6)$.

2.2 2-branes at $U(3)$ angles

We will consider an arbitrary number of 2-branes wrapped around various cycles. Choose a coordinate system so one of the cycles is the (y_1, y_2) cycle. According to Eq. 1 the 2-brane on this cycle imposes the projection $Q + \gamma^0\gamma^1\gamma^2\tilde{Q} = 0$. We write this as:

$$\gamma^0 Q = \gamma^1\gamma^2\tilde{Q} = -i(2\bar{\Gamma}^{\bar{1}}\Gamma^1 - 1)\tilde{Q} \tag{5}$$

Let us consider a collection of 2-branes rotated relative to this reference brane on the 6-torus. The ith brane is rotated by some $R_i \in SO(6)$ and imposes the supersymmetry projection:

$$\gamma^0 Q = (R_i\gamma)^1 (R_i\gamma)^2 \tilde{Q} \tag{6}$$

where R_i is in the fundamental representation of $SO(6)$. (Of course, on the 6-torus branes can only be rotated by a discrete subroup of $SO(6)$ if we require that they

[3] Our treatment may be extended to a general torus by using a vielbein as in [8].

[4] We use a mostly positive metric everywhere in this paper.

have finite winding number on all cycles.) Let $\mathcal{S}_{(R_i)}$ denote the corresponding rotation matrix in the spinor representation of $SO(6)$. Then the ith brane imposes the projection:

$$\gamma^0 Q = -i\mathcal{S}_{(R_i)}\,(2\Gamma^{\bar{1}}\Gamma^1 - 1)\,\mathcal{S}^\dagger_{(R_i)}\,\tilde{Q} \tag{7}$$

The Fock space elements $|n_1 n_2 n_3\rangle \otimes |n_4\rangle$ which form a basis for the spinors $\tilde{Q}$ are eigenstates of the Γ-matrix projections in Eq. 5 and Eq. 7: $-i(2\Gamma^{\bar{j}}\Gamma^j - 1)|n_1 n_2 n_3\rangle \otimes |n_4\rangle = i(1-2n_j)|n_1 n_2 n_3\rangle \otimes |n_4\rangle$. Therefore, there are simultaneous solutions of Eq. 5 and all the Eqs. 7 for each i, if there exist some $\tilde{Q}$ which are singlets under *all* the rotations: $\mathcal{S}_{(R_i)}\tilde{Q} = \mathcal{S}^\dagger_{(R_i)}\tilde{Q} = \tilde{Q}$.

Given such a collection of $R_i \in SO(6)$ that leave some $\tilde{Q}$ invariant, it is clear that any product of the R_i and their inverses will have the same property. It is easily seen that the set of such products will be a subgroup of $SO(6)$. The problem of finding supersymmetric relative rotations is therefore reduced to one of finding subgroups of $SO(6)$ that leave some $\tilde{Q}$ invariant. As discussed in the previous subsection, $\tilde{Q}$ transforms in the 8-dimensional spinor representation of $SO(6)$. The largest subgroup of $SO(6)$ under which spinors transform as singlets is $SU(3)$ with the decomposition $\mathbf{8} \to \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}$. This tells us that in general an arbitrary collection of branes that are related by $SU(3)$ rotations is supersymmetric. In fact, the branes can be related by $U(3)$ rotations and still be supersymmetric because the $U(1)$ factor in $U(3) = SU(3) \times U(1)$ cancels between $\mathcal{S}_{(R_i)}$ and $\mathcal{S}^\dagger_{(R_i)}$ in Eq. 7. In the special case when only two branes are present, the relative rotation can always be represented as an element of $SO(4)$ in the four dimensions spanned by the branes. In that case supersymmetry is preserved when the rotation is in an $SU(2)$ subgroup of $SO(4)$ as discussed in [7, 8]. However, when three or more branes are present they can explore all six dimensions of the torus and the present analysis applies.

We can determine the amount of supersymmetry surviving the presence of $U(3)$ rotated D2-branes by looking for $U(3)$ invariant spinors $\tilde{Q}$. Given the reference configuration Eq. 5 and the Fock basis discussed above, it is readily shown that the $U(3)$-invariant spinors are $\tilde{Q} = \{|000\rangle \otimes |n_4\rangle, |111\rangle \otimes |n_4\rangle\}$ where $n_4 = \{0, 1\}$. These four solutions give the equivalent of $N = 1, d = 4$ supersymmetry. (We will give a detailed explicit example that illustrates this general discussion in Sec. 2.5.)

Note that we can have an *arbitrary* number of D2-branes rotated at arbitrary $U(3)$ angles on the torus. In the special case that the rotations are in an $SU(2)$ subgroup of $U(3)$, we have enhanced supersymmetry since there are more spinors that are invariant under these rotations. Under $SU(2)$, the 8-dimensional spinor of $SO(6)$ decomposes as $\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{1} + \mathbf{1} + \mathbf{1}$. Recalling that the 16-component $SO(8)$ spinor $\tilde{Q}$ has a Fock space expansion $|n_1, n_2, n_3\rangle \otimes |n_4\rangle$ where only the $|n_2, n_2, n_3\rangle$ tranforms under $SO(6)$ we conclude that there are eight $SU(2)$ invariant solutions for $\tilde{Q}$. This gives the equivalent of $N = 2, d = 4$ supersymmetry, and reduces to the analysis of $SU(2)$ rotated 2-branes in [7, 8].

The significance of the $U(3)$ subgroup of $SO(6)$ is that it preserves some complex structure of the torus. Explicitly, there are coordinates (z_1, z_2, z_3) of the 6-torus that

transform in the $\mathbf{3}$ of $U(3)$ as:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \rightarrow R \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{8}$$

where R is in the fundamental representation of $U(3)$. The z_i are associated with complexified matrices Γ^i as discussed above, and these transform in the spinor representation as $(R\Gamma)^i \rightarrow \mathcal{S}\Gamma^i\mathcal{S}^\dagger$. In the preceding analysis we have essentially found that a system of 2-branes that are wrapped on arbitrary $(1,1)$ cycles relative to a given complex structure will be supersymmetric. Essentially, the choice of a pair of supersymmetric branes at angles picks out a distinguished complex structure for the torus and any number of further branes can be wrapped on $(1,1)$ cycles relative to that complex structure. Since any two $(1,1)$ cycles are related by $U(3)$ rotations, this means that the branes are $U(3)$ rotated *relative* to each other.

We can also consider the effects of *global* rotations on the entire system of branes. It is clear that global rotations of all the branes together will not affect the analysis of supersymmetry. Global rotations that belong to $U(3) \subset SO(6)$ are already included in the class of configurations of branes at *relative* $U(3)$ angles. So only the remaining global $SO(6)/U(3)$ acts nontrivially to produce new configurations. Indeed, by definition, $SO(6)/U(3)$ rotates the nine $(1,1)$ cycles of the torus that are preserved by the $U(3)$ into the six $(2,0)$ and $(0,2)$ cycles. As discussed above, an arbitrary number of branes at relative $U(3)$ angles can be wrapped supersymmetrically along any $(1,1)$ cycles. Manifestly, the global $SO(6)/U(3)$ rotates these branes into the remaining $(0,2)$ and $(2,0)$ cycles of the torus. In Sec. 5.1 we will argue that the most general BPS configuration of 2-branes on a 6-torus can be constructed by applying these global $SO(6)/U(3)$ rotations to our 2-branes at $U(3)$ angles.

2.3 Adding D6-branes

An arbitrary number of D6-branes can be added to the system of $U(3)$-rotated D2-branes discussed above without breaking any additional supersymmetry. Recall that the $\pm$ sign in the projection condition Eq. 1 reflects the two possible orientations of a brane. The presence of the $U(3)$ rotated D2-branes selects the D6-brane orientation: only D6-branes with the orientation associated with the minus sign in Eq. 1 can be introduced without breaking supersymmetry. With this orientation the supersymmetry projection becomes:

$$\gamma^0 Q = -i(2\bar{\Gamma}^{\bar{1}}\Gamma^1 - 1)(2\bar{\Gamma}^{\bar{2}}\Gamma^2 - 1)(2\bar{\Gamma}^{\bar{3}}\Gamma^3 - 1)\tilde{Q} \tag{9}$$

The spinors $\tilde{Q} = \{|000\rangle, |111\rangle\}$ provide simultaneous solutions to all the $U(3)$-rotated D2-brane conditions Eq. 7. It is immediately recognized that they also solve the D6-brane condition Eq. (9), giving the same relation between Q and $\tilde{Q}$.

We will see later that the corresponding classical configurations are four dimensional black holes with finite area. It is clear from the construction here that such black holes can be interpreted as bound states of D6-branes with the $U(3)$-rotated 2-branes.

2.4 D0-branes and D4-branes

Given the configuration constructed above, we may produce a similar configuration involving only D0-branes and D4-branes by T-dualizing the entire T^6. This transformation inverts the volume of the manifold and exchanges D6-branes with D0-branes. T-duality also converts each of the D2-branes in the system into a D4-brane wrapped on the $(2,2)$-cycle orthogonal to the $(1,1)$-cycle on which the D2-brane is wrapped. It follows that an arbitrary number of D4-branes wrapped on T^6 will be supersymmetric if they are related by relative $U(3)$ rotations of the complex coordinates. For the reasons discussed above, these $U(3)$-rotated systems are the most general supersymmetric configurations of D4-branes on T^6.

2.5 Example: 2226 with branes at angles

In this subsection we will work out an explicit example of D2-branes at $U(3)$ angles in the presence of a D6-brane. Three D2-branes are arranged as follows: one along y_1 and y_2, another along $y_1 \cos\alpha - y_3 \sin\alpha$ and $y_2 \cos\alpha - y_4 \sin\alpha$, and the third one along $y_1 \cos\beta - y_5 \sin\beta$ and $y_2 \cos\beta - y_6 \sin\beta$. In this system the second and third branes are rotated relative to the first one by independent $SU(2) \subset U(3)$ rotations. However, $U(3)$ does not support several commuting $SU(2)$ subgroups; so clearly this configuration explores $U(3)$ in a nontrivial way although the individual branes are separately rotated only by $SU(2)$

The supersymmetry projections implied by the three D2-branes and the D6-brane are:

$$
\begin{aligned}
\gamma^0 Q &= \gamma^1\gamma^2\tilde{Q} \\
\gamma^0 Q &= \Big(\gamma^1\cos\alpha - \gamma^3\sin\alpha\Big)\Big(\gamma^2\cos\alpha - \gamma^4\sin\alpha\Big)\tilde{Q} \\
\gamma^0 Q &= \Big(\gamma^1\cos\beta - \gamma^5\sin\beta\Big)\Big(\gamma^2\cos\beta - \gamma^6\sin\beta\Big)\tilde{Q} \\
\gamma^0 Q &= -\gamma^1\cdots\gamma^6\tilde{Q}
\end{aligned}
$$

Defining as before the complexified Γ-matrices,the supersymmetry conditions can be written:

$$
\begin{aligned}
\gamma^0 Q &= -i(2\bar{\Gamma}^{\bar{1}}\Gamma^1 - 1)\tilde{Q} \\
\gamma^0 Q &= -i[(2\bar{\Gamma}^{\bar{1}}\Gamma^1 - 1)\cos^2\alpha + (2\bar{\Gamma}^{\bar{2}}\Gamma^2 - 1)\sin^2\alpha + 2\sin\alpha\,\cos\alpha(\Gamma^1\bar{\Gamma}^{\bar{2}} + \Gamma^2\bar{\Gamma}^{\bar{1}})]\tilde{Q} \\
\gamma^0 Q &= -i[(2\bar{\Gamma}^{\bar{1}}\Gamma^1 - 1)\cos^2\beta + (2\bar{\Gamma}^{\bar{3}}\Gamma^3 - 1)\sin^2\beta + 2\sin\beta\,\cos\beta(\Gamma^1\bar{\Gamma}^{\bar{3}} + \Gamma^3\bar{\Gamma}^{\bar{1}})]\tilde{Q} \\
\gamma^0 Q &= -i(2\bar{\Gamma}^{\bar{1}}\Gamma^1 - 1)(2\bar{\Gamma}^{\bar{2}}\Gamma^2 - 1)(2\bar{\Gamma}^{\bar{3}}\Gamma^3 - 1)\tilde{Q}
\end{aligned}
$$

This system of equations has two solutions: $\Gamma^\mu\tilde{Q} = 0$, for which all of them reduce to $\gamma^0 Q = i\tilde{Q}$; and $\bar{\Gamma}^{\bar{\mu}}\tilde{Q} = 0$, for which they become $\gamma^0 Q = -i\tilde{Q}$. Each of the two solutions imposes 4 projections; so the complete configuration preserves $2\times 1/16 = 1/8$ of the supersymmetry of the vacuum. As expected the preserved spinors can be written in the Fock basis as $\tilde{Q} = \{|000\rangle, |111\rangle\}$.

If we eliminate the D6-brane and the third D2-brane, we are left with two D2-branes at an SU(2) angle. This is the simplest non-trivial example of branes at angles.

It was analyzed in detail in Refs. [7, 8]. Another special case is $\alpha = \beta = \frac{1}{2}\pi$ where we find the more familiar configuration of three orthogonally intersecting D2-branes in a D6-brane background. This was analyzed in Ref. [17].

3 Classical Branes at Angles

In Sec. 2, we saw that the most general supersymmetric configuration of D2-branes on T^6 consists of a collection of branes at $U(3)$ rotations. In other words all the D2-branes are wrapped on $(1,1)$-cycles relative to some given complex structure. In this section we will write the corresponding classical solutions to the supergravity equations. Various special cases of our solutions and their M-theory interpretations have been discussed in [1] and subsequent publications. We consider the case where the asymptotic 6-torus is square and has unit moduli. The solutions are most easily written in terms of the complex geometry of the compact space and so we begin by defining notation.

Choosing complex coordinates $z^j = (x^{2j-1} + ix^{2j})/\sqrt{2}$, the Kähler form of the asymptotic torus is:

$$k = i \sum_{J=1}^{3} dz^J \wedge d\bar{z}^{\bar{J}} \tag{10}$$

The volume of the asymptotic torus is:

$$\mathrm{Vol}(T^6) = \int_{T^6} \mathrm{dVol} = \int_{T^6} \frac{k \wedge k \wedge k}{3!} \tag{11}$$

and can been set equal to 1 without loss of generality by taking the asymptotic moduli to be unity. Our conventions for the normalization of forms, wedge products and Hodge dual may be found in Appendix A.

Now consider a collection of 2-branes wrapped on a square 6-torus and let ω_j be the volume form corresponding to the $(1,1)$-cycle on which the jth brane is wrapped. In keeping with the notation of Sec. 2 we think of each brane as being $U(3)$ rotated with respect to a reference configuration on the $(z^1, \bar{z}^1)$ torus. This gives:

$$\omega_j = i\,(R_{(j)})^1_J\,(R^*_{(j)})^1_K\, dz^J \wedge d\bar{z}^{\bar{K}}. \tag{12}$$

Branes wrapped at angles on the torus in this way will act as sources for the geometry causing the moduli to flow between infinity and the position of the branes in the classical solution. Remarkably, it turns out that to understand how the geometry behaves it is sufficient to simply add the $(1,1)$-forms of each 2-brane, with coefficients that are harmonic functions on the non-compact transverse 4-space:

$$\omega = \sum_j X_j\, \omega_j \tag{13}$$

In general we can choose $X_j = P_j/|\vec{r} - \vec{r}_j|$ where $\vec{r}_j$ is the position of the jth brane and $\vec{r}$ is the coordinate vector in the noncompact space. When the $\vec{r}_j$ are all different from each other, the constituent branes of the solution are separated in the noncompact

space and can be easily distinguished. In our case, we will be principally interested in four-dimensional black hole solutions and so we will usually work explicitly with the one-center form:

$$X_j = \frac{P_j}{r} \tag{14}$$

where P_j is the charge carried by the jth 2-brane[5]. Note again that we are only choosing the one-center form for the discussion of four dimensional black holes. The proof of spacetime supersymmetry in Sec. 6.4, for example, goes through for the general multi-center form of X_j.

In the one-center case, after quantizing the charges P_j (see Sec. 5.2), ω is essentially an element of the integer cohomology of the torus that characterizes the ensemble of cycles wrapped by the branes. Note that there are many different collections of branes that will have that same ω when the harmonic functions are one center. For example, an arbitrary ω can be generated by a global $U(3)$ rotation of some form characterizing three orthogonal branes: $\omega = \sum_{i=1}^{3} \tilde{P}_i dz^i \wedge d\bar{z}^{\bar{i}}$. As we will see below, the spacetime solution for a collection of branes is completely characterized by ω and this implies that in the one-center case, the *same* spacetime solution is shared by many *different* microscopic configurations of branes. This reflects the fact that there are many different configurations of branes at angles that have the same asymptotically measured charges. The no-hair theorem leads us to expect that such different sets of branes at angles with the same asymptotic charges will have the same spacetime solutions in the one-center case. The way to tell these configurations apart in the spacetime sense is to separate the constituent branes in the noncompact space by introducing the multi-center form of the harmonic functions.[6]

It is natural to define the intersection numbers

$$C_{ij} = \frac{1}{\mathrm{Vol}(T^6)} \int_{T^6} k \wedge \omega_i \wedge \omega_j \tag{15}$$

$$C_{ijk} = \frac{1}{\mathrm{Vol}(T^6)} \int_{T^6} \omega_i \wedge \omega_j \wedge \omega_k . \tag{16}$$

The C_{ijk} is proportional to the number of points at which a T-dual collection of 4-branes intersect on the 6-torus. This connection will be used in Sec. 5 in computing the BPS degeneracy of the configurations constructed in this section. As we will see shortly, the classical solution corresponding to 2-branes at angles can be completely specified in terms of the Kähler form of the asymptotic torus k, the 2-brane-form ω, and the number of 6-branes.

3.1 The Classical Solution

A classical solution corresponding to a collection of 6-branes and 2-branes at angles on a 6-torus is completely described in terms of the metric, the dilaton, the RR 3-

[5] To the extent that we are interested only in cohomology (it is natural to consider a basis of minimal area cycles), ω is an element of $H^{(1,1)}(T^6, \mathcal{O})$, with $\mathcal{O}$ the appropriate space of harmonic functions. However, because of the quantization condition on the P_j's (see Sec. 5.2), this is essentially integer cohomology.

[6] We thank A. Tseytlin for discussions regarding these points.

form gauge field and the RR 7-form gauge field. To construct the solution, we need only the 2-form $k+\omega$. The solution in string metric is:

$$ds^2 = (F_2F_6)^{1/2}dx_\perp^2 + (F_2F_6)^{-1/2}\left[-dt^2 + (h_{\mu\bar\nu}\,dz^\mu\,d\bar z^{\bar\nu} + h_{\bar\mu\nu}\,d\bar z^{\bar\mu}\,dz^\nu)\right] \quad (17)$$

$$A_{(3)} = \frac{1}{F_2}\,dt\wedge K \qquad ; \qquad A_{(7)} = -\frac{1}{F_6}\,dt\wedge \mathrm{dVol} \quad (18)$$

$$e^{-2\Phi} = \sqrt{\frac{F_6^3}{F_2}} \quad (19)$$

where the 2-form K is:

$$K \equiv *\frac{(k+\omega)\wedge(k+\omega)}{2!} \quad (20)$$

and is simply proportional to the internal Kähler metric in the presence of 2-branes:

$$G = ig_{\mu\bar\nu}\,dz^\mu\wedge d\bar z^{\bar\nu} = \frac{i}{\sqrt{F_2F_6}}h_{\mu\bar\nu}\,dz^\mu\wedge d\bar z^{\bar\nu} \equiv \frac{K}{\sqrt{F_2F_6}} \quad (21)$$

The functions F_2 and F_6 have simple expressions:

$$F_2 = \frac{\int_{T^6}(k+\omega)^3}{3!\mathrm{Vol}(T^6)} = 1 + \sum_i X_i + \sum_{i<j} X_iX_j\,C_{ij} + \sum_{i<j<k} X_iX_jX_k\,C_{ijk} \quad (22)$$

$$F_6 = 1 + \frac{Q_6}{r} \quad (23)$$

Here $dx_\perp^2 = dx_7^2 + dx_8^2 + dx_9^2$ refers to the noncompact part of the space.

The solution has a very simple structure: the 2-brane gauge field $A_{(3)}$ is proportional to the Kähler form of the compact manifold. Moreover, the determinant of the metric of the compact space is related to the dilaton through:

$$e^{2\Phi} = \sqrt{\det\, g_{\mathrm{int}}} = \sqrt{\frac{F_2}{F_6^3}} \quad (24)$$

as we derive in Appendix A. These simplifying features are important for the proof of spacetime supersymmetry that we present in Sec. 4.

In Sec. 3.6 we show that when the branes are orthogonal to each other our construction reduces to the "harmonic function rule" for orthogonally intersecting branes [14]. We will also show that when only two 2-branes are present, and are rotated at a relative $SU(2)$ angle, the solution of [11] is reproduced. Some of the configurations of intersecting branes displayed here have also appeared before in [1] and subsequent publications.

To develop intuition about the solution consider the asymptotics of the gauge fields:

$$A_{(7)} \overset{r\to\infty}{\longrightarrow} -dt\wedge\frac{k^3}{3!} + \frac{Q_6}{r}\,dt\wedge\frac{k^3}{3!} \quad (25)$$

$$A_{(3)} \overset{r\to\infty}{\longrightarrow} dt\wedge k - \sum_j \frac{P_j}{r}\,dt\wedge\omega_j \quad (26)$$

(The asymptotics of $A_{(3)}$ are derived in Appendix A.) The leading terms in both of these expressions are closed forms and do not contribute to the field strengths $F_{(4)} = dA_{(3)}$ and $F_{(8)} = dA_{(7)}$. The asymptotic field strengths are therefore determined by the $1/r$ terms. The difference in the sign of these terms between $A_{(7)}$ and $A_{(3)}$ reflects the fact explained in Sec. 2 that only anti-6-branes can be embedded on the same torus as 2-branes at $U(3)$ angles. As discussed above, $k^3/3!$ is the volume form of the asymptotic torus and and so $(-Q_6)$ is the 6-brane charge. From Eq. (26) it is apparent that the asymptotic observer sees a collection of 2-branes carrying charges P_j and wrapped around the cycles ω_j. The physical charges measured by this observer relative to a canonical basis of 2-cycles at infinity will be displayed in Sec. 3.3

3.2 Black Hole Mass and Horizon Area

The classical solution presented in Eqs. (17)-(19) describes a black hole in the non-compact four dimensions. The mass and area of the black hole should be computed from the four dimensional Einstein metric. The Einstein metric and the string metric of Eq. (17) are related via the four dimensional dilaton which is:

$$e^{-2\Phi_4} = e^{-2\Phi}\sqrt{\det g_{int}} = \sqrt{\frac{F_6^3}{F_2}}\sqrt{\frac{F_2}{F_6^3}} = 1 \tag{27}$$

Here we used Eq. (24) for the determinant of the metric of the compact space. So the four dimensional Einstein metric is the same as the four dimensional string metric and is given by:

$$ds_4^2 = (F_2F_6)^{-1/2}(-dt^2) + (F_2F_6)^{1/2}(dr^2 + r^2 d\Omega^2) \tag{28}$$

This metric describes a black hole with horizon at $r = 0$.

The mass of the black hole can be read off from the behavior of the metric as $r \to \infty$:

$$ds_4^2 \stackrel{r\to\infty}{\longrightarrow} \left(1 - \frac{Q_6 + \sum_j P_j}{2r}\right)(-dt^2) + \left(1 + \frac{Q_6 + \sum_j P_j}{2r}\right)(dr^2 + r^2 d\Omega^2). \tag{29}$$

The mass M is:

$$4G_N M = Q_6 + \sum_j P_j. \tag{30}$$

The total mass is simply the sum of the constituent masses of the objects in the bound state since a single 6-brane will have mass $4G_N M = Q_6$ and each individual 2-brane has a mass $4G_N M = P_j$. Therefore the solutions constructed here are in fact *marginal* bound states in the sense that they have vanishing binding energy.

To compute the area of the horizon at $r = 0$, note that the area of a sphere of constant radius r in the metric Eq. (28) is $A = 4\pi r^2 (F_2F_6)^{1/2}$. Now $F_6 \simeq Q_6/r$ and:

$$\begin{aligned} F_2 \quad &\stackrel{r\to 0}{\longrightarrow} \quad \frac{\int_{T^6} \omega\wedge\omega\wedge\omega}{3!\mathrm{Vol}(T^6)} \\ &= \quad \frac{1}{r^3}\sum_{i<j<k} P_iP_jP_k \frac{\int_{T^6}\omega_i\wedge\omega_j\wedge\omega_k}{\mathrm{Vol}(T^6)} = \frac{1}{r^3}\sum_{i<j<k} P_iP_jP_k C_{ijk}. \end{aligned} \tag{31}$$

Using this equation we find that the area of the horizon is:

$$A = 4\pi (Q_6 \sum_{i<j<k} P_i P_j P_k C_{ijk})^{1/2}. \tag{32}$$

Written this way, the formula for the area is reminiscent of the area formulae for double extreme black holes in $N = 2$ string theory [18]. Indeed, the solution constructed in this section has relied purely on the Kähler nature of the compact geometry and we are in the process of generalizing our results to compactifications on Calabi-Yau 3-folds [15]. In Sec. 5 we will write the entropy of the black hole $S = A/4G_N$ in terms of the numbers of different kinds of branes and use this to count the microscopic degeneracy of the corresponding D-brane configuration.

3.3 Canonical Charge Matrix

An observer at infinity looking at the classical solutions constructed in this section would not *a priori* decompose the configuration into constituent 2-branes. Rather, the asymptotic observer would measure a 3-form gauge field arising from 2-brane charges associated with all 15 cycles of the 6-torus. Considering the asymptotics of the 3-form field in Eq. (26) we are led to define the charge vector:

$$q_\alpha = r \int_{\Omega_\alpha} \omega = \sum_i P_i \int_{\Omega_\alpha} \omega_i \tag{33}$$

where the Ω_α are a basis of 2-cycles for the 6-torus. In general there are 15 parameters q_α. However, we know from the discussion of supersymmetry in Sec. 2 that the ω_i are all $(1,1)$ forms relative to some choice of complex structure. So after a suitable choice of basis cycles, only 9 of the q_α are non-vanishing. These charges transform in the $\mathbf{3} \otimes \bar{\mathbf{3}}$ representation of $U(3)$ which rotates the complex structure and it is therefore natural to assemble them into a matrix $q_{a\bar{b}}$. Below we will express the physical properties of the solution - its mass and horizon area - in terms of this canonical charge matrix. Note that in addition to this nine parameter matrix of charges, the solution is characterized by the 6-brane charge giving a 10 parameter family of black holes. In fact, configurations of 2-branes and 6-branes on T^6 are characterized in general by 16 charges. The remaining 6 parameters arise from global $SO(6)/U(3)$ rotations that deform the complex structure of the solution. These parameters are in one-to-one correspondence with the $(0,2)$ and $(2,0)$ cycles. According to the analysis of Sec. 2 it is not possible to add additional 2-branes that lie along these cycles consistently with supersymmetry; so these charges can only be turned on by global rotations of the entire solution.

We construct the canonical charge matrix using a basis of 2-forms $\Omega^{a\bar{b}}$ that are dual to the $(1,1)$ cycles. We can then define a set of projection coefficients $\alpha_{ia\bar{b}}$ that relate the cycles ω_i on which the 2-branes are wrapped to the basis cycles:

$$\omega_i = \sum_{a\bar{b}} \alpha_{ia\bar{b}} \Omega^{a\bar{b}} \tag{34}$$

In terms of these projection coefficients, the canonical charge matrix is:

$$q_{a\bar{b}} = \sum_i P_i \alpha_{ia\bar{b}} \tag{35}$$

and the 2-form ω characterizing the collection of 2-branes is $\omega = \sum_{a\bar{b}} (q_{a\bar{b}}/r)\Omega^{a\bar{b}}$. Our branes at angles construction can be used to generate an *arbitrary* charge matrix.

To express the mass and area of the black hole in terms of the charge matrix it is convenient to choose the $\mathbf{3} \otimes \bar{\mathbf{3}}$ basis $\Omega^{a\bar{b}} = i dz^a \wedge d\bar{z}^{\bar{b}}$. In terms of this basis the intersection form C_{ijk} is:

$$C_{ijk} = \frac{\int_{T^6} \omega_i \wedge \omega_j \wedge \omega_k}{\mathrm{Vol}(T^6)} = \alpha_{ia_1\bar{b}_1}\, \alpha_{ja_2\bar{b}_2}\, \alpha_{ka_3\bar{b}_3} \epsilon^{a_1a_2a_3} \epsilon^{\bar{b}_1\bar{b}_2\bar{b}_3} \tag{36}$$

This means that:

$$\sum_{i<j<k} P_i P_j P_k C_{ijk} = q_{a_1\bar{b}_1}\, q_{a_2\bar{b}_2}\, q_{a_3\bar{b}_3}\, \frac{\epsilon^{a_1a_2a_3} \epsilon^{\bar{b}_1\bar{b}_2\bar{b}_3}}{3!} = \det q \tag{37}$$

So we can write the area of the black holes as:

$$A = 4\pi \sqrt{Q_6 \det q} \tag{38}$$

The fact that the area formula is an invariant of global $U(3)$ rotations is a consequence of duality. Indeed, as will be discussed in Sec. 5.1, Eq. (38) arises precisely as a quartic invariant of the E(7,7)-symmetric central charge matrix of Type II string compactified on a 6-torus. The mass of the black hole is another invariant of the charge matrix: the trace. To see this, note that $Tr(q) = \sum_{ia} P_i \alpha_{ia\bar{a}}$. But in the $\mathbf{3} \otimes \bar{\mathbf{3}}$ basis,

$$\sum_a \alpha_{ia\bar{a}} = \frac{\int_{T^6} \omega_i \wedge k \wedge k}{2!\mathrm{Vol}(T^6)} = 1 \tag{39}$$

The last equality follows because ω_i is the volume form of a $(1,1)$ cycle and k is invariant under $U(3)$ rotations that preserve the complex structure. (In fact, we have already used Eq. (39) in writing Eq. (22).) So we find that:

$$4G_N M = Q_6 + \sum_j P_j = Q_6 + \mathrm{Tr}(q) \tag{40}$$

The last equation expresses the total mass in terms of asymptotic charges: this is the BPS saturation formula.

3.4 T-duality: 4440

The configurations of 2-branes and 6-branes constructed above can be converted into systems of 4-branes at angles bound to 0-branes by T-dualizing along every cycle of the 6-torus. Below we will T-dualize the solution of Sec. 3.1 to arrive at a solution for 4-branes at angles in the presence of a 0-brane. In this formulation, *T-duality of all six directions amounts to Poincaré duality followed by inversion of the internal metric.* As we will see, the resulting solution is remarkably simple, compared even to that of the 2- and 6-branes.

Gauge Fields: A 2-brane wrapped on the cycle ω_i dualizes to a 4-brane wrapped on $\omega_{(4)i} = *\omega_i$. Similarly, the 6-brane wrapped on the the 6-cycle $k \wedge k \wedge k/3!$ dualizes to a 0-brane characterized by a 0-form $\omega_0 = *(k^3/3!)$. We conclude that T-duality of the 6-torus acts as Poincaré duality on the gauge fields of the solution. This gives 5-form and 1-form fields in the dualized solutions:

$$A_{(5)} = \frac{1}{F_2} dt \wedge *K = \frac{1}{F_2} dt \wedge \frac{(k+\omega)\wedge(k+\omega)}{2!} \tag{41}$$

$$A_{(1)} = \frac{1}{F_6} dt \tag{42}$$

where we used the identity $** = 1$ for $*$ acting on any (p,p) form in our conventions.

Metric: The metric of the 6-torus is inverted by T-duality. If $G = i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\bar{\nu}}$ is the Kähler form associated with metric g, the Kähler form associated with the inverse metric g^{-1} is:[7]

$$G^{-1} \equiv *(G \wedge G)/(2! \det g_{\mu\bar{\nu}}) \tag{43}$$

In our solution the Kähler form of the compact metric is:

$$G = \frac{K}{\sqrt{F_2 F_6}} = * \frac{(k+\omega)\wedge(k+\omega)}{2! \sqrt{F_2 F_6}} \tag{44}$$

Writing $\kappa = (k+\omega)$ and recalling the definition of F_2 in Eq. (22), we see that $K/F_2 \equiv \kappa^{-1}$ in the notation introduced in Eq. (43). So $K^{-1} \equiv \kappa/F_2$ and the Kähler form of the compact space is transformed by T-duality into:

$$G' = G^{-1} = \kappa \sqrt{\frac{F_6}{F_2}} = (k+\omega)\sqrt{\frac{F_6}{F_2}} \tag{45}$$

This also inverts the determinant of the metric of the internal space giving $\det g_{int} \xrightarrow{T} F_6^3/F_2$.

Dilaton: Under T-duality the dilaton transforms as as $2\Phi' = 2\Phi - \ln \det g_{int}$ [19]. The determinant of the metric is computed in Appendix A to be $\det g_{int} = F_2/F_6^3$. So we find that the dilaton is transformed by T-duality into:

$$e^{-2\Phi'} = e^{-2\Phi} \det g_{int} = \sqrt{\frac{F_2}{F_6^3}} \tag{46}$$

Since systems of 2-branes and 6-branes on a 6-torus are dual to systems of 4-branes and 0-branes, we see that the most general supersymmetric state of 4-branes on T^6 is constructed by wrapping an arbitrary number of 4-branes on $(2,2)$ cycles relative to some complex structure. Each brane is characterized by a $(2,2)$ form $\omega_{(4)i}$

[7] With our conventions for the Hodge dual, $*(G \wedge G)/2 = (i/2)\,[g_{\mu_1\bar{\nu}_1}\, g_{\mu_2\bar{\nu}_2}]\, \epsilon^{\mu_1\mu_2}{}_{\alpha}\, \epsilon^{\bar{\nu}_1\bar{\nu}_2}{}_{\bar{\beta}}\, dz^{\alpha} \wedge d\bar{z}^{\bar{\beta}}$. But this precisely computes the matrix of minors used in computing the inverse of g. Dividing by the determinant of g gives the Kähler form of the inverse metric.

and the collection of branes is characterized by the form: $\omega_{(4)} = \sum_j X_j \,\omega_{(4)j}$ where $X_j = P_j/r$ is a harmonic function on the non-compact space. In terms of the form ω characterizing the dual 2-branes we have:

$$\omega = *\omega_{(4)} = \sum_j X_j \, * \omega_{(4)j} = \sum_j X_j \,\omega_j \tag{47}$$

The intersection numbers defined in Sec. 3 can be introduced again for the collection of 4-branes in terms of the dual 2-forms.

We can now use the above discussion of T-duality to summarize the classical solution of 4-branes at angles in the presence of 0-branes. Define as in Sec. 3.1 the quantities:

$$\begin{aligned} F_4(r) &= \frac{\int_{T^6}(k + *\omega_{(4)})^3}{3!\mathrm{Vol}(T^6)} = 1 + \sum_i X_i + \sum_{i<j} X_i X_j \, C_{ij} + \sum_{i<j<k} X_i X_j X_k \, C_{ijk} && (48) \\ F_0 &= 1 + \frac{Q_0}{r} && (49) \end{aligned}$$

Relabeling F_2 as F_4 and F_6 as F_0 in the T-dual system above, the solution is given by:

$$\begin{aligned} ds^2 &= (F_4 F_0)^{1/2} dx_\perp^2 + (F_4F_0)^{-1/2}\left[-dt^2 + (h_{\mu\bar{\nu}}\, dz^\mu \; d\bar{z}^{\bar{\nu}} + h_{\bar{\mu}\nu}\, d\bar{z}^{\bar{\mu}} \; dz^\nu)\right] && (50) \\ G_4 &= i g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^{\bar{\nu}} = i\sqrt{\frac{1}{F_0F_4}} h_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu \equiv K_4\sqrt{\frac{F_0}{F_4}} && (51) \\ K_4 &\equiv (k + *\omega_{(4)}) && (52) \\ e^{-2\Phi} &= \sqrt{\frac{F_4}{F_0^3}} && (53) \\ A_{(5)} &= \frac{1}{F_4}\, dt \wedge \frac{K_4 \wedge K_4}{2!} \qquad ; \qquad A_{(1)} = -\frac{1}{F_0}\, dt && (54) \end{aligned}$$

As before, the metric of the compact space has been specified in terms of its Kähler form G_4. The four dimensional Einstein metric is left invariant by T-duality and so these 4440 configurations describe four-dimensional black holes just like the 2226 systems described earlier. The mass and the area continue to be given by Eq. (30) and Eq. (32).

3.5 Mystical Comments

The extremely simple form of the classical solution for 4- and 0-branes is striking. The internal Kähler form is simply a sum of the asymptotic Kähler form and the 2-form dual to the collection of 4-branes. Here we make a few remarks that hint at the underlying geometric structure.

A 4-brane on T^6 has (complex) codimension one, and so it defines a *divisor* of the torus. Associated with any such divisor is a line bundle, whose first Chern class is the 2-form dual to the divisor. Here the Chern class of the direct sum of line bundles is

making its appearance in the Kähler form, a result familiar from symplectic geometry. These considerations, as well as the simple action of T-duality on the solutions point to a simple way of including the effects of fluxes of various kinds in the brane solutions, as we will discuss in a future article [15].

3.6 Example and Relation to Known Results

To build intuition for our solution it is useful to work out the details for the example with three 2-branes and one 6-brane in Sec. 2.5. Recall that this had three sets of branes, rotated by different $SU(2)$ subgroups of $U(3)$. To construct the classical solution in Eq. (17)-Eq. (19) we need only compute the function $F_2(r)$ and the Kähler form of the 6-torus metric. To do this, we begin by constructing the form ω characterizing the ensemble of 2-branes:

$$\begin{aligned}\omega = \textstyle\sum_j X_j \omega_j &= X_1\ i dz^1 \wedge d\bar{z}^{\bar{1}} + X_2\ i(\cos\alpha dz^1 - \sin\alpha dz^2) \wedge (\cos\alpha d\bar{z}^{\bar{1}} - \sin\alpha d\bar{z}^{\bar{2}}) \\ &\quad + X_3\ i(\cos\beta dz^1 - \sin\beta dz^3) \wedge (\cos\beta d\bar{z}^{\bar{1}} - \sin\beta d\bar{z}^{\bar{3}}) \end{aligned} \tag{55}$$

Here $X_j = P_j/r$. We can easily compute the intersection numbers C_{ij} and C_{ijk} and find that F_2 is given by:

$$\begin{aligned} F_2 &= 1 + \sum_j X_j + X_1 X_2 \sin^2\alpha + X_1 X_3 \sin^2\beta \\ &\quad + X_2 X_3 (\sin^2\alpha + \sin^2\beta\cos^2\alpha) + X_1 X_2 X_3 \sin^2\alpha \sin^2\beta \end{aligned} \tag{56}$$

The Kähler form of the compact space is $G = K/\sqrt{F_2 F_6}$, where

$$K = \frac{*(k+\omega)^2}{2!} = k + \sum_j X_j (k - \omega_j) + \sum_{i<j} X_i X_j\ *(\omega_i \wedge \omega_j) \tag{57}$$

A small computation shows $*(\omega_1 \wedge \omega_2) = \sin^2\alpha\ \Omega^{3\bar{3}}$, $*(\omega_1 \wedge \omega_3) = \sin^2\beta\ \Omega^{2\bar{2}}$ and:

$$\begin{aligned} *(\omega_2 \wedge \omega_3) &= \Big[c_\alpha^2 s_\beta^2\ \Omega^{2\bar{2}} + s_\alpha^2 c_\beta^2\ \Omega^{3\bar{3}} + s_\alpha^2 s_\beta^2\ \Omega^{1\bar{1}} \\ &\quad + c_\beta s_\beta s_\alpha^2\ (\Omega^{3\bar{1}} + \Omega^{1\bar{3}}) + c_\alpha s_\alpha s_\beta^2\ (\Omega^{2\bar{1}} + \Omega^{1\bar{2}}) + c_\beta c_\alpha s_\beta s_\alpha\ (\Omega^{3\bar{2}} + \Omega^{2\bar{3}})\Big] \end{aligned} \tag{58}$$

where we are using $\Omega^{a\bar{b}} \equiv i dz^a \wedge d\bar{z}^{\bar{b}}$, and $s_\alpha \equiv \sin\alpha$, etc. All quantities in Eq. (17)-Eq. (19) are specified in terms of F_2 and K, and so the above formulae completely specify the spacetime solution corresponding to the three 2-branes and the 6-brane.

Orthogonal Branes: When $\alpha = \beta = \pi/2$ we have a configuration of three orthogonal 2-branes wrapped on the (12),(34) and (56) cycles of the torus in the presence of a 6-brane. In that case F_2 factorizes into a product of harmonic functions:

$$F_2 = 1 + \textstyle\sum_j X_j + \sum_{i<j} X_i X_j + X_1 X_2 X_3 = (1+X_1)(1+X_2)(1+X_3) \tag{59}$$

$$e^{-2\Phi} = \left(\frac{F_6^3}{(1+X_1)(1+X_2)(1+X_3)}\right)^{1/2} \tag{60}$$

The Kähler form of the 6-torus metric is given by

$$G = \frac{i}{\sqrt{F_6}}\left[dz^1\wedge d\bar{z}^{\bar{1}}\left(\frac{(1+X_2)(1+X_3)}{(1+X_1)}\right)^{1/2} + dz^2\wedge d\bar{z}^{\bar{2}}\left(\frac{(1+X_1)(1+X_3)}{(1+X_2)}\right)^{1/2} + dz^3\wedge d\bar{z}^{\bar{3}}\left(\frac{(1+X_1)(1+X_2)}{(1+X_3)}\right)^{1/2}\right] \quad (61)$$

and the gauge fields are:

$$A_{(3)} = i\,dt\wedge\left[\frac{dz^1\wedge d\bar{z}^{\bar{1}}}{(1+X_1)} + i\frac{dz^2\wedge d\bar{z}^{\bar{2}}}{(1+X_2)} + i\frac{dz^3\wedge d\bar{z}^{\bar{3}}}{(1+X_3)}\right] \quad ; \quad A_{(7)} = \frac{-1}{F_6}\,dt\wedge\frac{k^3}{3!} \quad (62)$$

This solution coincides exactly with the "harmonic function rule" that governs orthogonally intersecting branes as discussed in [14].

$SU(2)$ **angles:** By eliminating the third 2-brane and the 6-brane we arrive at the system discussed in [7, 8] - a pair of 2-branes at a relative $SU(2)$ rotation. Classical solutions corresponding to such configurations were given in [11] and discussed in [13]. In this case we find:

$$F_2 = 1 + X_1 + X_2 + X_1X_2\sin^2\alpha \quad (63)$$

$$K = k + \textstyle\sum_j X_j(k-\omega_j) + iX_1X_2\sin^2\alpha\ dz^3\wedge d\bar{z}^{\bar{3}} \quad (64)$$

The Kähler form of the metric of the compact space $G = K/(F_2F_6)^{1/2}$ and the dilaton coincide with the metric and dilaton in the solution of [11]. To compare our gauge field with the one in [11] it is helpful to modify it by adding a closed form which does not affect the field strength:

$$A'_{(3)} = A_{(3)} + dt\wedge\frac{k^3}{3!} = \frac{dt}{F_2}\wedge K + dt\wedge\frac{k^3}{3!} \quad (65)$$

This gauge field coincides exactly with the one given in [11, 13].[8]

Mass, Area and Charges: The mass of the example 2226 configuration is given by Eq. (30) as $4G_NM = Q_6 + \sum_j P_j$. Using Eq. (32) for the area of the black hole with C_{ijk} as in the computation of F_2 in Eq. (56) we find that:

$$A = 4\pi\sqrt{Q_6(P_1P_2P_3)\sin^2\alpha\,\sin^2\beta} \quad (66)$$

This is reminiscent of the area formula for the NS-NS black hole generating solution presented in [20]. The canonical charge matrix defined relative to the $\mathbf{3}\oplus\bar{\mathbf{3}}$ basis $dz^i\wedge d\bar{z}^{\bar{j}}$ is:

$$q = \begin{pmatrix} P_1 + P_2\cos^2\alpha + P_3\cos^2\beta & -P_2\sin\alpha\cos\alpha & -P_3\sin\beta\cos\beta \\ -P_2\sin\alpha\cos\alpha & P_2\sin^2\alpha & 0 \\ -P_3\sin\beta\cos\beta & 0 & P_3\sin^2\beta \end{pmatrix} \quad (67)$$

It is manifest in terms of this matrix that $4GM = Q_6 + Tr(q)$ and $A = 4\pi\sqrt{Q_6\det q}$.

[8] Note that the authors of [11] are working with branes wrapped on different cycles and with charges that are opposite to ours. Our results for the special case of 2-branes at $SU(2)$ angles agree after some trivial relabelling of coordinates.

T-duality: The 2226 example in this section transforms under T-duality of the 6-torus into three 4-branes and a 0-brane at relative angles. The 4-form characterizing the collection of 4-branes is $\omega_{(4)} = *\omega$. Following Eq. (50)-Eq. (54), the solution is completely characterized by F_4 and K_4. Now $F_4 = F_2$ from the definition of F_4 and the fact that $*\omega_{(4)} = **\omega = \omega$. Furthermore $K_4 = (k + \omega)$. It is instructive to verify that for orthogonal branes this reproduces the "harmonic function rule" of [14]. When $\alpha = \beta = \pi/2$ we have 4-branes wrapped on the (3456),(1256) and (1234) cycles of the 6-torus in the presence of a 0-brane. We find that the dilaton and gauge field are given by:

$$e^{-2\Phi} = \left(\frac{(1+X_1)(1+X_2)(1+X_2)}{F_0^3}\right)^{1/2} \tag{68}$$

$$A_{(5)} = dt \wedge \left[\frac{\Omega^{1\bar{1}} \wedge \Omega^{2\bar{2}}}{(1+X_3)} + \frac{\Omega^{1\bar{1}} \wedge \Omega^{3\bar{3}}}{(1+X_2)} + \frac{\Omega^{2\bar{2}} \wedge \Omega^{3\bar{3}}}{(1+X_1)}\right] \tag{69}$$

The Kähler form of the torus is given by:

$$G_4 = \sqrt{F_0}\left[\Omega^{1\bar{1}}\left(\frac{(1+X_1)}{(1+X_2)(1+X_3)}\right)^{1/2} + \Omega^{2\bar{2}}\left(\frac{(1+X_2)}{(1+X_1)(1+X_3)}\right)^{1/2} + \right.$$
$$\left. +\Omega^{3\bar{3}}\left(\frac{(1+X_3)}{(1+X_1)(1+X_2)}\right)^{1/2}\right] \tag{70}$$

This verifies the "harmonic function rule" for orthogonal 4-branes and 0-branes.

4 Spacetime Supersymmetry

We have argued in Section 2 that the D-brane configurations at arbitrary $U(3)$ angles are supersymmetric. In Section 3, we exhibited an ansatz for the corresponding classical solution of the supergravity equations of motion. In the present section, we prove directly that this ansatz preserves the same supersymmetries as the corresponding D–brane configuration. It is expected that the spacetime supersymmetry and the Bianchi identities together imply the equations of motion, and we verify this explicitly in Appendix B. Thus the Killing spinor equations are essentially the "square root" of the equations of motion.

4.1 The *Ansatz* and the SUSY variations

The solutions discussed in this paper are completely determined by the Kähler metric $g_{\mu\bar{\nu}}$ of the compact space. We will see that the existence of supersymmetric solutions to the supergravity equations depends essentially only on the Kähler property of the internal metric, and on simple relationships between that metric and the other fields. As such, the solutions presented in this paper for Type IIA branes on a torus are in fact special cases of a much more general situation. Indeed, the analysis of this

section appears to carry over almost unchanged to branes compactified on Calabi-Yau 3-folds [15].

We begin with a solution describing 2-branes only. In that case, the other non-vanishing fields are given in terms of $g_{\mu\bar{\nu}}$ as:

$$
\begin{aligned}
g_{00} &= -F^{-1/2} && (71)\\
g_{ij} &= \delta_{ij} F^{1/2} && (72)\\
A_3 &= F^{-1/2} dt \wedge i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} && (73)\\
e^{2\Phi} &= F^{1/2} && (74)
\end{aligned}
$$

where

$$\sqrt{\det\, g_{\rm int}} = \det\, g_{\mu\bar{\nu}} = \sqrt{F}\ . \tag{75}$$

We will use the notation $(I, J, \cdots)$ for general spacetime indices, with 0 as the time index, and introduce holomorphic internal indices $(\mu, \nu \cdots)$ (and their complex conjugates), and external indices $(i, j, \cdots)$. The internal metric and, by implication, all the other fields, depend on the external coordinates x^i only. In this section $z^{\mu} = x^{2\mu-1} + i x^{2\mu}$; so the flat space metric has $\eta_{z\bar{z}} = \frac{1}{2}$ and $\eta^{z\bar{z}} = 2$.[9] As discussed in Section 3, Eqs.(71)–(74) can describe an arbitrary number of 2-branes at angles. The inclusion of an additional 6-brane is needed for construction of black holes. This extension turns out to be very simple and we will consider it in Sec. 4.4. The case of 4-branes and 0-branes, as discussed earlier, is related by T-duality to the analysis of this section.

The supersymmetry variations of bosonic fields vanish automatically in backgrounds that contain no fermionic fields. However, an explicit calculation is needed to show that the variation of the fermionic fields also vanish. In Einstein frame, these variations are available in [21]; in string frame they are:

$$
\begin{aligned}
\sqrt{2}\,\delta\lambda &= [\frac{1}{2}\partial_I\Phi\ \Gamma^I\Gamma^{11} - \frac{3}{16}\ e^{\Phi}F_{IJ}\Gamma^{IJ} - \frac{i}{192}\ e^{\Phi}F_{IJKL}\Gamma^{IJKL}]\epsilon && (76)\\
\delta\psi_I &= [(\partial_I + \frac{1}{4}\omega_{JK,I}\Gamma^{JK}) + \frac{1}{8}e^{\Phi}F_{JK}(\frac{1}{2!}{\Gamma_I}^{JK} - {\delta_I}^J\Gamma^K)\Gamma^{11} + \\
&+ \frac{i}{8}e^{\Phi}F_{JKLM}(\frac{1}{4!}{\Gamma_I}^{JKLM} - \frac{1}{3!}{\delta_I}^J\Gamma^{KLM})\Gamma^{11}]\epsilon && (77)
\end{aligned}
$$

The dilatino and gravitino fields are denoted λ and ψ_I, respectively, and $\omega_{JK,I}$ is the spin connection. We are interested in configurations carrying only Ramond-Ramond charges and so the NS 2-form B has been taken to vanish. For the moment, we will also not include 0-, 4- or 6-branes, so the 2-form field strength may be set to zero. The fermionic parameters of the two supersymmetries have been combined into a single field ϵ that is Majorana but not Weyl.

[9] This differs from the conventions used elsewhere in this paper. The F in this section is denoted F_2 elsewhere.

4.2 Preliminaries

It is worthwhile to introduce some notation and carry out various preliminary calculations before returning to the explicit verification that the supersymmetry variations indeed vanish on the stated solution.

Zehnbein: In Eqs. (76)-(77) the supersymmetry variations are written using standard curved space indices. In order to compute the spin connection that appears in the supersymmetry variations, it is useful to work in a local orthonormal frame. To this end, it is convenient to introduce a zehnbein $e_{\hat{I}}{}^{I}$ satisfying:

$$e_{\hat{I}}{}^{I} e_{\hat{J}}{}^{J} g_{IJ} = \eta_{\hat{I}\hat{J}}, \tag{78}$$

and their inverses $e_{J}{}^{\hat{I}}$. Here the indices of the local orthonormal frame are denoted with hats. Explicit expressions for the zehnbein can be found by solving Eq. (78). Given the form of the metric above, we clearly have:

$$e_0{}^{\hat{0}} = \sqrt{-g_{00}} = F^{-1/4} \tag{79}$$

$$e_i{}^{\hat{\jmath}} = \delta_i^j F^{1/4} \tag{80}$$

In the familiar examples of parallel or orthogonally intersecting branes the internal metric is also diagonal, but in the general case considered here it is not. Although the internal components of the zehnbein can still be found, the expressions are unwieldy and not particularly illuminating. Their explicit form will not be needed.

Spin Connection: The components of the spin connection with indices along the transverse space are conveniently calculated from the zehnbein. They are:

$$\omega_{\hat{\jmath}\hat{k},i} = \frac{1}{4F}[\delta_{ij}\partial_k - \delta_{ik}\partial_j]F \tag{81}$$

$$\omega_{\hat{\jmath}\hat{0},0} = -\frac{1}{4F^{3/2}}\partial_j F \tag{82}$$

On the other hand, the components of the spin-connection with internal indices are simple in their curved space form. The only non-zero components are:

$$\omega_{j\nu,\bar{\mu}} = -\frac{1}{2}\partial_j g_{\bar{\mu}\nu} \tag{83}$$

This simple form arises because there harmonic functions appearing in the metric depend only on coordinates transverse to the internal space.

Field Strength: Next, we consider the 4-form field strength. The non-vanishing components may be computed directly from the potential (Eq. (73)):

$$F_{i0\mu\bar{\nu}} = i\partial_i(F^{-1/2} g_{\mu\bar{\nu}}) \tag{84}$$

There will be repeated need for the trace of this expression over the internal indices. This takes a particularly simple form:

$$
\begin{aligned}
F_{i0\mu\bar{\nu}} g^{\mu\bar{\nu}} &= i[3\partial_i(F^{-1/2}) + F^{-1/2} g^{\mu\bar{\nu}} \partial_i g_{\mu\bar{\nu}})] && (85) \\
&= 2i\partial_i F^{-1/2} && (86)
\end{aligned}
$$

To obtain the last line, we have used Eq. (75) in the form

$$
g^{\mu\bar{\nu}} \partial_i g_{\mu\bar{\nu}} = \partial_i \ln \det g_{\mu\bar{\nu}} = \frac{1}{2F} \partial_i F \tag{87}
$$

Note that, unlike other equations in this section, Eq. (85) is specific to an internal space that has three complex dimensions. In other cases Eq. (86) must be established anew. (It is straightforward to do so for internal spaces that have two complex dimensions and we anticipate that other cases follow similarly.)

Supersymmetry: We know that the supersymmetry is partially broken, and so some constraint will be placed on the spinor ϵ. We will find solutions to Eqs. (76),(77) provided either of the conditions

$$
\Gamma^{\bar{\mu}} \epsilon = 0 \tag{88}
$$

or

$$
\Gamma^{\mu} \epsilon = 0 \tag{89}
$$

are satisfied. Note, for example, that the first condition is equivalent to

$$
\Gamma^{\hat{\bar{\mu}}} \epsilon = 0 \tag{90}
$$

since the zehnbein respects the complex structure. These conditions are entirely equivalent to the conditions found in the D-brane analysis (see for example, Sec. 2.5),[10] and so we are satisfied that the classical p-brane solutions are describing precisely the same D-brane configurations. Below, we will choose the constraint Eq. 88; the analysis of the states satisfying the second constraint is essentially the complex conjugate of the one presented here.

4.3 Proof of Supersymmetry

With these preparations, the explicit calculations are quite simple. In what follows we will examine the variations of the dilatino and the external and internal components of the gravitino.

[10]That is, Eqs. (90) and (93) are equivalent to the solution in terms of Fock space states $|000\rangle$ and $|111\rangle$ in Sec. 2 .

Dilatino: The dilatino variation is

$$\sqrt{2}\delta\lambda = [\frac{1}{2}\partial_I\Phi\Gamma^I\Gamma^{11} - \frac{i}{192}\ e^{\Phi}\ F_{IJKL}\Gamma^{IJKL}]\epsilon = 0 \tag{91}$$

The projection conditions on the spinor Eq. (88) and the identity Eq. (86) give:

$$\frac{1}{4!}F_{IJKL}\Gamma^{IJKL}\epsilon = F_{i0\mu\bar{\nu}}\Gamma^{i0\mu\bar{\nu}}\epsilon = -F_{i0\mu\bar{\nu}}g^{\mu\bar{\nu}}\Gamma^{i0}\epsilon = -2i\partial_i F^{-1/2}\Gamma^{i0}\epsilon \tag{92}$$

Also using $e^{\Phi} = F^{1/4}$ we find

$$\Gamma^{11}\epsilon = -\Gamma^{\hat{0}}\epsilon\ . \tag{93}$$

This condition is simply a projection on the spinor. Note that the curved index on Γ^0 was converted to a flat index using $e_{\hat{0}}{}^0$. This is appropriate because it is $\Gamma^{\hat{0}}$ which squares to -1.

Temporal Gravitino: The variation of the time component of the gravitino is

$$\delta\psi_0 = [(\partial_0 + \frac{1}{4}\omega_{JK,0}\Gamma^{JK}) + \frac{i}{8}e^{\Phi}F_{JKLM}(\frac{1}{4!}\Gamma_0{}^{JKLM} - \frac{1}{3!}\delta_0{}^J\Gamma^{KLM})\Gamma^{11}]\epsilon = 0 \tag{94}$$

We will assume that the spinor is static. Moreover, the indices on the field strength must include a 0; so the first of the terms with gamma-matrices vanishes, by antisymmetrization. Multiplication of Eq. (92) with Γ^0 gives:

$$-\frac{1}{3!}F_{JKLM}\delta_0{}^J\Gamma^{KLM}\epsilon = -2i\partial_i F^{-1/2}\Gamma^i\epsilon \tag{95}$$

Using the spin connection Eq. (82) we recover the same projection condition on the spinor that was found from the dilatino variation (Eq. (93)).

External Gravitini: The variations of the external components of the gravitini are:

$$\delta\psi_i = [(\partial_i + \frac{1}{4}\omega_{jk,i}\Gamma^{jk}) + \frac{i}{8}e^{\Phi}F_{j0\mu\bar{\nu}}(\Gamma_i{}^{j0\mu\bar{\nu}} - \delta_i{}^j\Gamma^{0\mu\bar{\nu}})\Gamma^{11}]\epsilon = 0 \tag{96}$$

In this case we need the identity:

$$F_{j0\mu\bar{\nu}}(\Gamma_i{}^{j0\mu\bar{\nu}} - \delta_i{}^j\Gamma^{0\mu\bar{\nu}})\Gamma^{11}\epsilon = -F_{j0\mu\bar{\nu}}g^{\mu\bar{\nu}}(\Gamma_i{}^j - \delta_i{}^j)\Gamma^0\Gamma^{11}\epsilon \tag{97}$$

$$= -2i\partial_j F^{-\frac{1}{2}}(\Gamma_i{}^j - \delta_i{}^j)\Gamma^0\Gamma^{11}\epsilon \tag{98}$$

The spin connection $\omega_{jk,i}$ (Eq. (81)) cancels the term proportional to $\Gamma_i{}^j$ when the projection Eq. (93) is imposed on the spinor. This leaves:

$$(\partial_i + \frac{1}{8F}\partial_i F)\epsilon = 0 \tag{99}$$

The solution is:

$$\epsilon = F^{-1/8}\epsilon_\infty = e^{-\Phi/2}\epsilon_\infty\ . \tag{100}$$

It is a generic property of supersymmetric solutions to supergravity that the supersymmetry parameters depend on spacetime coordinates. The dependence on the radial coordinate found here is the same as the one that occurs in the single p-brane solutions, if both are expressed in terms of the dilaton. A Weyl rescaled spinor can be defined that is constant throughout spacetime.

Internal Gravitini: The final equations to check are the variations of the internal components of the gravitino. Given the projection (88), the components $\delta\psi_\mu$ vanish trivially. We also need to check that

$$\delta\psi_{\bar{\mu}} = [\frac{1}{4}\omega_{JK,\bar{\mu}}\Gamma^{JK} + \frac{i}{8}e^{\Phi}F_{JKLM}(\frac{1}{4!}\Gamma_{\bar{\mu}}{}^{JKLM} - \frac{1}{3!}\delta_{\bar{\mu}}{}^{J}\Gamma^{KLM})\Gamma^{11}]\epsilon = 0 \tag{101}$$

The identity

$$F_{JKLM}(\frac{1}{4!}\Gamma_{\bar{\mu}}{}^{JKLM} - \frac{1}{3!}\delta_{\bar{\mu}}{}^{J}\Gamma^{KLM})\Gamma^{11}\epsilon = F_{i0\bar{\rho}\nu}(g^{\bar{\rho}\nu}\Gamma_{\bar{\mu}}{}^{i0} - 2\delta^{\bar{\rho}}_{\bar{\mu}}\Gamma^{i0\nu})\Gamma^{11} \tag{102}$$

$$= 2iF^{-\frac{1}{2}}\partial_i g_{\bar{\mu}\nu}\Gamma^{i0\nu}\Gamma^{11}\epsilon \tag{103}$$

and the spin connection with internal indices (Eq. (83)) reduces the equation to the projection Eq. (93), as before. Note that this equation relates individual components of the gauge field and the internal metric, in contrast to previous conditions that only involved the determinant of the internal metric. This tensorial structure forces the property of our solutions that the gauge field is proportional to the Kähler form of the compact space.

This completes the verification of spacetime supersymmetry for 2-branes at angles.

4.4 The addition of 6-branes

In applications to black holes, the metric describing 2-branes at angles must be augmented with a 6-brane. This generalization is straightforward and does not introduce issues that are not already present for the discussion of orthogonally intersecting branes in [14]. The discussion will therefore be brief.

The six-branes couple to an 8-form field strength which is related to the 2-form field strength by Hodge duality. This gives a new term in each supersymmetry variation. In the presence of a 6-brane the components of the metric are multiplied by a conformal factor. This results in an additional term in each of the spin connections. Similarly a term dependent on the 6-brane is added to the logarithm of the dilaton. In this way several terms are added to each spinor variation. The required cancellations follow from the fact that the 6-brane is supersymmetric when there are no 2-branes present, since the supersymmetry variations are linear in field strengths.

There is only one issue that requires detailed attention: the presence of the 6-brane changes the dilaton and the metric components in a multiplicative fashion; so it must be shown that such factors cancel, leaving the original supersymmetry conditions on the 2-brane system intact. It is straightforward to show that the relevant precise condition is:

$$[e^{\Phi}e^{0}_{\hat{0}}e^{\mu}_{\hat{\mu}}e^{\nu}_{\hat{\nu}}]_{6-\text{brane only}} = F_6^{3/4}(F_6^{-1/4})^3 = 1 \tag{104}$$

Similarly the cancellation needed to show that the presence of the 2-branes does not affect the 6-brane supersymmetry variations is:

$$[e^{\Phi}e^{0}_{\hat{0}}\det{}^{-1}g_{\hat{\mu}\hat{\nu}}]_{2-\text{branes only}} = F^{1/4}\ F^{1/4}\ F^{-1/2} = 1 \tag{105}$$

The full 2-2-2-6 configuration is therefore supersymmetric, as advertized.

5 Four Dimensional Black Hole Entropy

In previous sections we have constructed two T-dual 16-parameter classes of four dimensional black holes, one containing 2-branes and 6-branes and another containing 4-branes and 0-branes. In this section we will show that the entropy of these black holes can be understood microscopically as arising from the degeneracy of the corresponding bound state of D-branes. In general, an extremal black hole in IIA string theory on a 6-torus has a 56 dimensional charge vector and the entropy of the black hole is constructed from the charges in terms of the quartic invariant of the E(7,7) duality group. In Sec. 5.1 we will discuss where our black holes belong in this parameter space and how the representation theory of $E(7,7)$ predicts exactly the entropy formula that we have found, convincing us that we have correctly identified two full 16-parameter subspaces of the spectrum of four dimensional black holes. In Sec. 5.2 we rewrite the area of the black hole solutions of this paper in terms of quantized charges to arrive at a formula for the entropy of the black holes that is expressed purely in terms of the number of different kinds of branes. In Sec. 5.3 this representation of the entropy is matched to the microscopic degeneracy derived by counting the number of bound states of the corresponding D-brane configuration.

5.1 Black Holes and U-duality

According to the no-hair theorem black holes are completely characterized by their mass, angular momentum, and $U(1)$ charges. For extremal black holes in four dimensions the angular momentum vanishes and the mass is given by the BPS formula; so the $U(1)$ charges are the only independent parameters. In Type II supergravity compactified on T^6, the $U(1)$ charges transform in a **56** dimensional representation of the $E(7,7)$ duality symmetry[11]. There are also scalar fields (moduli) present, but they are not independent parameters: a given $U(1)$ charge is always multiplied by a certain combination of moduli. It is in fact these "dressed" charges, *i.e.* charges with moduli absorbed in them, that appear in our supergravity solutions. The moduli parametrize the coset $E(7,7)/SU(8)$; so $SU(8)$ is the duality group that transforms the dressed charges, but leaves the moduli invariant. Since the moduli parametrize inequivalent vacua, this reduction of symmetry is simply the phenomenon of spontaneous symmetry breaking.

The charges transform in the antisymmetric tensor representation of $SU(8)$. They can be represented schematically in terms of the central charge matrix of the supersymmetry algebra:

$$\mathcal{Z}_{AB} = \frac{1}{4}\begin{pmatrix} (Q_R + iP_R)_{ab} & \mathbf{R}_{ab} \\ -\mathbf{R}_{ba} & (Q_L + iP_L)_{ab} \end{pmatrix} \tag{106}$$

Here $Q_{R,L}$ and $P_{R,L}$ are the NS charge vectors, written in the spinor representation, and the 4×4 matrix $\mathbf{R}$ contains the 32 RR charges. In this paper we are interested in configurations that do not couple to NS-fields. Note that the matrix $\mathbf{R}$ transforms

[11] Some useful references for this subsection are [22, 23, 24].

in a complex $(\mathbf{4}, \mathbf{4})$ representation of the $SU(4)_R \times SU(4)_L$ subgroup of $SU(8)$. The geometric $SO(6) \sim SU(4)$ global rotation group on the compact space is embedded into the $SU(4)_R \times SU(4)_L$ in such a way that $\mathbf{R}$ transforms under it as $2(\mathbf{4} \otimes \bar{\mathbf{4}}) = \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{15} \oplus \mathbf{1}$. Here the the two $\mathbf{1}$s are clearly the 0-branes and 6-branes and the $\mathbf{15}$s are the 2-branes and 4-branes. It is convenient to work in an $SO(6)$ basis, in which the explicit form of the matrix $\mathbf{R}$ is:

$$\mathbf{R}_{a\bar{b}} = (-Q_6 + iQ_0)\delta_{a\bar{b}} + (\frac{1}{2!} r_{ij} + \frac{i}{4!}\epsilon_{ijklmn} r^{klmn})\Gamma^{ij}_{a\bar{b}} \tag{107}$$

where $-Q_6$ and Q_0 are the D6- and D0-brane charges, r_{ij} and r^{klmn} are the projections of the D2- and D4-brane charge matrices on to the 15 different 2-cycles and 4-cycles respectively, and the matrices $\Gamma^{ij}_{a\bar{b}}$ are generators of $SO(6)$ in the spinor representation[12] The $\Gamma^{ij}_{a\bar{b}}$ can be interpreted as Clebsch-Gordon coefficients that realize the equivalence $SU(4) \simeq SO(6)$. This decomposition is clearly consistent with the interpretation of the charge matrix in terms of 6-branes, 2-branes, 4-branes and 0-branes.

Consider an arbitrary 2-brane charge tensor $r_{a\bar{b}} = r_{ij}\Gamma^{ij}_{a\bar{b}}/2$. Since r is an element of the Lie algebra of $SO(6) \sim SU(4)$ it transforms in the adjoint representation of the group. From Sec. 3.3 we know that our branes at *relative* $U(3)$ angles can be used to generate a family of charge matrices q that fill out a $U(3)$ subalgebra of the space of $SU(4)$ Lie algebra elements comprising the general charge matrix r. It is readily shown that the orbit of a $U(3)$ subalgebra of $SU(4)$ under the action of the quotient group $SU(4)/U(3)$ covers the entire $SU(4)$ algebra.[13] The configurations of branes constructed in this paper were wrapped on $(1,1)$ cycles relative to some complex structure. Since $U(3)$ is the group that preserves the complex structure, the $SU(4)/U(3)$ global rotations generating the general charge matrix from our configurations are simply understood as complex structure deformations associated with the $(2,0)$ and $(0,2)$ cycles. This shows that, after allowing for global rotations, our solutions realize the most general charge configuration.

We will now write the area formula for a black hole with arbitrary charges by exploiting the fact that the Einstein metric is invariant under duality, so that expressions for the mass and the area must be invariant functions of the charges. In fact the entropy $S = A/4G_N$ of a black hole has the much stronger property that it does not depend on moduli and so must by expressible in terms of the integral quantized charges as opposed to the physical "dressed" charges appearing in the supergravity solutions [3]. This can be proven using supersymmetry [25]. The entropy is therefore invariant under the full non-compact $E(7,7)$ [26]. Generalizing from the simplest examples of orthogonally intersecting branes, it follows that the area of the most general extremal four dimensional black hole in $N = 8$ supergravity is $A = 4\pi\sqrt{J_4}$ where J_4 is the unique quartic invariant of $E(7,7)$:

$$J_4 = Tr\ (\mathcal{Z}^\dagger \mathcal{Z})^2 - \frac{1}{4}(Tr\ \mathcal{Z}^\dagger \mathcal{Z})^2 + \frac{1}{96}(\epsilon_{ABCDEFGH}\mathcal{Z}^{AB}\mathcal{Z}^{CD}\mathcal{Z}^{EF}\mathcal{Z}^{GH} + c.c) \tag{108}$$

[12] The sign of Q_6 has been chosen so that $(-Q_6)$ is the 6-brane charge, consistently with Sec. 3.

[13] To show this we use the fact that if L is a $U(3)$ subalgebra, then there are no generators w of $SU(4)/U(3)$ with the property that $[w, l] \in L$ for all $l \in L$.

As written in Eq. 107, the components of $R_{a\bar{b}}$ are not directly associated with branes wrapped on different cycles since the Γ matrices mix up the different components of r_{ij}. Consequently, it is easier to interpret the charges in the $SO(8)$ formalism where we rewrite the central charge matrix as:

$$\frac{1}{\sqrt{2}}(x_{ij} + iy_{ij}) = -\frac{1}{4}\mathcal{Z}_{AB}(\rho_{ij})^{AB} \tag{109}$$

Here $(\rho_{ij})^{AB}$ are generators of $SO(8)$. The $SO(8)$ form of the central charge matrix is particularly useful because individual branes correspond to specific components of the 8×8 matrices x_{ij} and y_{ij}. The precise correspondence between branes wrapped on specific cycles and particular components of the x and y matrices can be found by explicitly writing out the transformation into $SO(8)$ basis. In the case of our $U(3)$ family of D2-brane configurations it is convenient to introduce complexified coordinates to describe the first six of the eight $SO(8)$ indices, while keeping the indices 7 and 8 unchanged. In this notation it can be shown that for our configurations the non-vanishing components of the central charge matrix are $y_{a\bar{b}} = \frac{1}{\sqrt{2}}q_{a\bar{b}}$ and $y_{78} = -\frac{1}{\sqrt{2}}Q_6$ where $q_{a\bar{b}}$ are the canonical charges defined in Sec. 3.3.

The fact that in our solution the x_{ij} vanish for some choice of complex structure reflects a simplification that is not generic[14]. For the $SU(8)$ form of the charge matrix in Eq. 107 the analogous statement is that the central charge matrix can be chosen real. In fact, in $N = 2$ and $N = 4$ supergravity the central charge can always be chosen real after applying suitable dualities, because in these cases the compact part of the duality group is $U(N)$. On the other hand, for $N = 8$ supergravity the compact part of the duality group is $SU(8)$, as we have seen, so in general the central charge matrix has an invariant phase [20, 24]. It follows that there are configurations in which the central charge cannot be chosen to be real. Although very general, our configurations do not capture this feature of $N = 8$ supergravity.

We can now calculate from group theory the area of a black hole carrying general 2-brane and 6-brane charges. In the $SO(8)$ formalism the quartic invariant is:

$$-J_4 = x^{ij}y_{jk}x^{kl}y_{li} - \frac{1}{4}x^{ij}y_{ij}x^{kl}y_{kl} + \frac{1}{96}\epsilon_{ijklmnop}(x^{ij}x^{kl}x^{mn}x^{op} + y^{ij}y^{kl}y^{mn}y^{op}) \tag{110}$$

For our configurations only the last term of this expression in nonvanishing:

$$J_4 = Q_6 \det q \tag{111}$$

This gives a black hole area of $A = 4\pi\sqrt{Q_6 \det q}$ in agreement with the explicit calculation of Sec. 3.3. Note that the derivation of the entropy given in this section is independent of the explicit solution: it relies only on supersymmetry and duality invariance.

[14] Acting on all the branes with a global $SU(4)/U(3)$ rotation that does not respect the complex structure would turn on some x_{ij}.

5.2 Quantized Charges

The first step towards a microscopic understanding of the black hole entropy is to express it in terms of the quantized charges. We need to understand how the physical charges P_i and Q_6 appearing in the harmonic functions governing the solution are related to the numbers of branes wrapped on different cycles. As before we will assume that the asymptotic torus is square and has unit moduli as in the solutions presented in Sec. 3. We will work with the solution containing 2-branes and 6-branes - the discussion of the solution containing 4-branes and 0-branes proceeds analogously. Recall that we defined the canonical charges $q_{ja\bar{b}}$ induced on the (1,1)-cycles of the torus by a given 2-brane wrapped on the cycle ω_j via:

$$P_j \omega_j = \sum_{a\bar{b}} P_j \alpha_{ja\bar{b}}\, \Omega^{a\bar{b}} \equiv \sum_{a\bar{b}} q_{ja\bar{b}}\, i dz^a \wedge d\bar{z}^{\bar{b}} \tag{112}$$

Adapting the formulae in the Appendix of [27] for branes compactified on T^6 it is easy to show the quantization condition for the canonical 2-brane charges and the 6-brane charge:

$$q_{ja\bar{b}} = \frac{l_s g}{4\pi} n_{ja\bar{b}} \qquad ; \qquad Q_6 = \frac{l_s g}{4\pi} N_6 \tag{113}$$

Here $n_{ja\bar{b}}$ counts the number of times the jth 2-brane wraps the $(a\bar{b})$ cycle and N_6 counts the number of times the 6-brane wraps the entire torus. The string length is $l_s = 2\pi\sqrt{\alpha'}$. This quantization condition leads us to write:

$$P_j \omega_j = \frac{l_s g}{4\pi} \sum_{a\bar{b}} n_{ja\bar{b}} \Omega^{a\bar{b}} \equiv \frac{l_s g}{4\pi} M_j \sum_{a\bar{b}} m_{ja\bar{b}} \Omega^{a\bar{b}} \equiv \frac{l_s g}{4\pi} M_j v_j \tag{114}$$

where we have defined M_j to be the greatest common factor of the $n_{ja\bar{b}}$. Now M_j is the integral number of branes wrapping the cycle ω_j and v_j is the element of the *integral* cohomology of the torus that characterizes the cycle. (Recall that ω_j was a *volume* form and as such not a member of the integral cohomology.) We then define:

$$N_{ijk} = \frac{1}{\mathrm{Vol}(T^6)} \int_{T^6} v_i \wedge v_j \wedge v_k \tag{115}$$

Recall that T-duality of the 6-torus converts the 2-branes into 4-branes wrapped on the cycles $*\omega_i$. The forms $*v_i$ characterize these dual 4-branes in integral cohomology. Three 4-branes on a 6-torus generically intersect at a point and the N_{ijk} are integers counting the number of intersection points. Using the quantized charges we find that:

$$\sum_{i<j<k} P_i P_j P_k C_{ijk} = \left(\frac{l_s g}{4\pi}\right)^3 \sum_{i<j<k} (M_i M_j M_k) N_{ijk} \tag{116}$$

Putting this together with the 6-brane quantization condition we can rewrite the area formula as:

$$A = 4\pi \left(\frac{l_s g}{4\pi}\right)^2 \sqrt{N_6 \sum_{i<j<k} (M_i M_j M_k) N_{ijk}} \tag{117}$$

Finally, the entropy of the black hole is given by $S = A/4G_N$ and the Newton coupling constant is $G_N = g^2\alpha'/8 = g^2 l_s^2/32\pi^2$. This gives:

$$S = 2\pi\sqrt{N_6 \sum_{i<j<k} (M_i M_j M_k) N_{ijk}} \tag{118}$$

In the next section we will explain the origin of this entropy microscopically in terms of the degeneracy of the corresponding bound state of D-branes.

Example: To understand these results intuitively it is instructive to explicitly evaluate the entropy in terms of quantized charges in the example of Sec. 3.6. We will work in the T-dual picture where there are three 4-branes and examine the meaning of the quantization conditions to show how the intersection number of branes arises in the entropy formula. The first brane is wrapped on the coordinates (y_3, y_4, y_5, y_6), the second is on $((s_\alpha y_1 + c_\alpha y_3), (s_\alpha y_2 + c_\alpha y_4), y_5, y_6)$ and the third is on $((s_\alpha y_1 + c_\alpha y_5), (s_\alpha y_2 + c_\alpha y_6), y_2, y_3)$. (We have adopted the notation of Sec. 3.6 and taken $\omega_{(4)i} = *\omega_i$ where the ω_i characterize the 2-branes in Eq. (55)). The area of the resulting black hole, as evaluated in Eq. (66), is:

$$A = 4\pi\sqrt{Q_0 P_1 P_2 P_3 \sin^2\alpha \sin^2\beta} \tag{119}$$

(We have relabelled Q_6 as Q_0 after T-duality.) Since the 4-branes must wrap a finite number of times around the torus, the angles α and β are quantized as $\tan_\alpha = q_\alpha/p_\alpha$ and $\tan_\beta = q_\beta/p_\beta$ where each q, p pair are relatively prime integers. Projecting the quantization conditions discussed above on to the cycles defined by the branes we write the P_i as:

$$P_i = \frac{l_s g}{4\pi} M_i A_i \tag{120}$$

where M_i is the wrapping number of the ith brane and A_i is its dimensionless area ($A_1 = 1$, $A_2 = p_\alpha^2 + q_\alpha^2$, and $A_3 = p_\beta^2 + q_\beta^2$). This makes good sense since each brane has a fixed charge density and so the total physical charge of a brane should be proportional to its area. So we see that the quantization condition for the canonical charges $q_{ia\bar{b}}$ produces the correct quantization of the charges of the angled branes. Inserting these quantized charges into the area Eq. (119) and dividing by $4G_N$ gives the entropy:

$$S = 2\pi\sqrt{N_0(M_1 M_2 M_3) q_\alpha^2 q_\beta^2} \tag{121}$$

We now want to argue that $q_\alpha^2 q_\beta^2$ counts the number of intersection points of the three 4-branes on the 6-torus. Consider a string wound on the (0,1) cycle of a 2-torus and another string on a (q,p) cycle. It is clear that the strings intersect q times. In our case we have one 4-brane on the (3456) cycle and another one on the ([13][24](56)) cycle. (The square brackets indicate angling on the corresponding torus.) The first brane is on a (0,1) cycle on the (13) and (24) 2-tori. The second brane is on a (q_α, p_α) cycle on both these tori. So it is clear that the two 4-branes intersect in q_α^2 places on the (1234) torus and each intersection has two-dimensional extent along the (56) cycle. Each of these intersection manifolds lying on the (56) cycle is intersected in q_β^2

places by the third 4-brane leading to $q_\alpha^2 q_\beta^2$ mutual intersections of the three 4-branes. So the factor of $\sin^2\alpha \sin^2\beta$ in Eq. (119), when multiplied by the factors of area in the quantization condition for angled branes, precisely reproduces the intersection number of the three 4-branes as discussed more abstractly above.

5.3 Counting the States of The Black Hole

Counting the states of the black hole is easiest in the picture with 4-branes and 0-branes where the entropy formula is $S = 2\pi\sqrt{N_0 \sum_{i<j<k}(M_i M_j M_k) N_{ijk}}$. In this case the analysis is exactly parallel to the one described in [17, 28]. More recently, a similar discussion has appeared for certain black holes in $N = 2$ string theory in [29, 30]. The counting is aided by the M-theory perspective where 0-brane charge arises as momentum in the 11th dimension and 4-branes are the dimensional reduction of M-theory 5-branes wrapped on the 11th dimension. Three intersecting 4-branes arise in 11 dimensions as three 5-branes intersecting along a line and the 0-branes that we are interested in arise as momentum along one direction of this line.

Large N_0: The argument is simplest when N_0, the number of 0-branes, greatly exceeds the number of 4-brane intersections. The leading contribution to the black hole entropy arises in this case from the number of different ways in which a total 0-brane charge of N_0 can be distributed between the 4-brane intersections. Since the momentum in the 11th dimension can come in integral multiples, the charge of 0-branes bound to the mutual intersection of three 4-branes can also come in integral multiples. We now *assume* that the 0-branes bound to the 4-branes, or momentum modes along the intersection string along the 11th dimension give rise to states that appear as B bosonic and F fermionic species with an associated central charge of $c = B + (1/2)F = 6$. (See Sec. 5.4 for the origin of this assumption in an effective string description of the black hole entropy.) Since there are a total of $N_{int} = \sum_{i<j<k}(M_i M_j M_k) N_{ijk}$ intersection points of 4-branes, the problem is very simply to count the number of ways of distributing B bosonic and F fermion 0-branes that come in integrally charged varieties amongst N_{int} intersections. From the 11-dimensional perspective, we want to distribute a total momentum of N_0 carried by B bosonic and F fermionic modes amongst N_{int} strings. As discussed in [17, 28], both of these problems are identical to the computation of the density of states at level N_0 of a string with central charge $c_{\text{eff}} = cN_{int} = 6N_{int}$. For large N_0, the level density is $d(N_0) = \exp 2\pi\sqrt{N_0 c/6}$ giving an entropy of [31]:

$$S = \ln d(N_0) = 2\pi \sqrt{N_0 \sum_{i,j,k} (M_i M_j M_k) N_{ijk}} \tag{122}$$

This exactly reproduces Eq. (118), the entropy formula for the black hole!

General N_0: The above counting of states only works in the limit when N_0 greatly exceeds the number of 4-brane intersections, because that is the regime of validity

of the asymptotic level density formula that gives rise to the entropy. In the general case where the number of 0-branes is comparable to the number of 4-brane intersections, the leading contribution to the black hole entropy comes from states in which the 4-brane charges arise from *multiply-wrapped* branes rather than from multiple *singly-wrapped* branes. From the M-theory perspective, the multiple wrappings of the 5-branes increase the length of the effective string along which the momentum propagates, changing the quantization condition for the momentum and hence the 0-brane charge. As discussed in [32, 33], this corrects the level density asymptotics in exactly the right way to make the state counting discussed above applicable to the large charge classical black hole regime. The application of [32, 33] to precisely this context of 4-branes and 0-branes is discussed in [17, 28].

5.4 Effective String Description

The discussion in Sec. 5.3 relied on the assumption that the momentum modes on the mutual intersection line of three M-theory 5-branes have B bosonic and F fermionic degrees of freedom so that $c = B + (1/2)F = 6$. Some arguments in favor of this assumption were presented in [17, 28]. Certainly the assumption seems to succeed in accounting for the entropy of black holes in a wide variety of cases in Type II strings compactified on both tori [17, 28] and on Calabi-Yau 3-folds [29, 30]. This suggests that we should turn the argument around and use the matching with the entropy of black holes to derive the physics of the intersection manifold of 5-branes in M-theory.

Three 5-branes intersecting along a line in 11 dimensions have a (0,4) supersymmetry on that line. This means that only left-moving momentum can be added to the string without breaking supersymmetry as indeed we found in Sec. 2 after T-dualizing the resulting 0-branes in 10 dimensions into 6-branes. While the mutual intersection line of the 5-branes is wrapped along the 11th dimension, the other dimensions of the 5-branes are wrapped on a T^6 in our solution. Consider making the T^6 small while leaving the 11th dimension large. The resulting effective string should be described by a $(0,4)$ supersymmetric sigma model on the orbifold target:

$$\mathcal{M} = \frac{(T^6)^{M_i M_j M_k}}{S(M_i M_j M_k)} \tag{123}$$

where the M_i are the quantized charges of the three 5-branes and we orbifold by the symmetric group $S(M_i M_j M_k)$ to account for symmetry under exchange of the 5-branes.[15] The asymptotic degeneracy of BPS states of this effective string will reproduce the entropy formula Eq. (118) for the case where only three 5-branes are present, following the work of [6, 35, 34].

In our case, each triplet of 5-branes intersects in N_{ijk} locations giving rise to N_{ijk} effective strings. So from the M-theory perspective, our solutions are described in

[15]Note that one might have naively supposed that that appropriate orbifold group would be $S(M_i)S(M_j)S(M_k)$ which would account for the exchange symmetry of each kind of 5-brane. However, the analysis of [2, 34] indicates that the appropriate group is $S(M_1M_2M_3)$. Indeed, this is the orbifold that is consistent with T-duality.

the small torus limit by $N_{tot} = \sum_{i<j<k} N_{ijk}$ effective strings, each propagating on an orbifold like Eq. (123). The appropriate total effective conformal field theories have central charges that are the sum of contributions from many effective strings, and the resulting degeneracy exactly matches our Eq. (118).

6 Conclusion

In this paper we have shown that the most general supersymmetric state of 2-branes on a 6-torus is constructed using an arbitrary number of branes at relative $U(3)$ angles, acted upon by global $SO(6)/U(3)$ rotations. After addition of 6-branes, these configurations account for a 16-parameter subspace of the spectrum of BPS states of IIA string theory on T^6. T-duality of the torus converts these states into the most general BPS configurations of 4-branes and 0-branes on a 6-torus. We have constructed the corresponding solutions to the supergravity equations and verified explicitly that they solve the Killing spinor equations. The spacetime solutions are remarkably simple when expressed in terms of the complex geometry of the compact space. Very little in our analysis has relied on toroidal structure of the compact space - the relevant properties are that the compact space is Kähler and the gauge fields are proportional to the Kähler form. In fact, we expect our results to go through with only slight modifications for compactifications on Calabi-Yau 3-folds, which we are in the process of verifying [15]. The geometric structure of our solutions suggests natural generalizations that would provide the spacetime solutions for arbitrary BPS bound states of Type II solitons. We are in the process of investigating these generalizations as well. The configurations constructed in this paper can be interpreted as black holes in four dimensions. We computed the thermodynamic entropies of these black holes and showed that they can be interpreted microscopically in terms of the bound state degeneracy of the corresponding collection of D-branes. Our calculations agree beautifully with the analysis of four dimensional black hole entropy in terms of the quartic invariant of the $E(7,7)$ duality group.

Acknowledgments: We would like to thank C. Callan and M. Cvetič for numerous interesting discussions and G. Papadopoulos and E. Sharpe for helpful comments. We have also had useful correspondence with A. Tseytlin. V.B. was supported partly by DOE and NSF grants DE-FG02-91-ER40671 and NSF-Phy-91-18167 and also by the Harvard Society of Fellows. F.L. was supported in part by DOE grant AC02-76-ERO-3071.

A Useful Properties of the Solutions

In this appendix we will derive the asymptotics of the 3-form gauge field and then compute the determinant of the metric of internal 6-torus. Throughout this paper we are using the following definition of the Hodge dual of (p,q) forms:

$$* \left(dz^{\mu_1} \wedge \cdots dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \cdots d\bar{z}^{\nu_q}\right) = \quad i^{p+q+1} \frac{1}{(3-p)!\,(3-q)!} \epsilon^{\mu_1 \cdots \mu_p}{}_{\alpha_1 \cdots \alpha_{3-p}} \epsilon^{\nu_1 \cdots \nu_q}{}_{\beta_1 \cdots \beta_{3-q}} \times$$

$$dz^{\alpha_1} \wedge \cdots dz^{\alpha_{3-p}} \wedge d\bar{z}^{\beta_1} \wedge \cdots d\bar{z}^{\beta_{3-q}} \quad (124)$$

where $\epsilon_{123} = 1$ and the indices are raised and lowered using the flat metric of the asymptotic torus.

3-form Asymptotics: The 3-form gauge field is given by $A_{(3)} = (1/F_2) dt \wedge K$. We want to extract the leading r dependence at large r. As $r \to \infty$, $1/F_2 \to 1 - \sum_j X_j$. We also have $K = *(k+\omega)^2/2 \sim k + 2\sum_j X_j * (k \wedge \omega_j)$ for large r. In the conventions of Sec. 2 and Sec. 3, ω_j is a $U(3)$ rotation of the $(1,1)$ form $dz^1 \wedge d\bar{z}^1$. The Kähler form k of the asymptotic torus is invariant under such $U(3)$ rotations that preserve the complex structure, and it is easy to use this and the definition of the Hodge dual to show that:

$$* (k \wedge \omega_j) = (k - \omega_j) \quad (125)$$

This gives the asymptotics:

$$A_{(3)} \stackrel{r\to\infty}{\longrightarrow} dt \wedge k \; - \; \sum_j \frac{P_j}{r} (dt \wedge \omega_j) \quad (126)$$

where we used the definition $X_j = P_j/r$ for the harmonic function associated with the jth brane.

Determinant of 6-torus metric: The metric of the compact space g_{int} is most readily specified in terms of its Kähler form:

$$G = \frac{1}{\sqrt{F_2 F_6}} K \quad (127)$$

where $K = *\kappa^2/2$ and $F_2 = \int_{T^6} \kappa \wedge \kappa \wedge \kappa / 3! \, \mathrm{Vol}(T^6)$ with $\kappa = (k + \omega)$. Writing $G = i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$, while the metric of the compact space is $ds^2 = g_{\mu\bar{\nu}} dz^\mu d\bar{z}^\nu + g_{\bar{\mu}\nu} d\bar{z}^\mu dz^\nu$, we find:

$$\sqrt{\det g_{int}} = \frac{1}{3! \, \mathrm{Vol}(T^6)} \int_{T^6} G \wedge G \wedge G = \frac{1}{3! \, \mathrm{Vol}(T^6) \sqrt{F_2^3 F_6^3}} \int_{T^6} K \wedge K \wedge K \quad (128)$$

To compute $K \wedge K \wedge K$ it is helpful to write $\kappa = (k + \omega) = i f_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$ so that $\kappa \wedge \kappa \wedge \kappa / 3! = \det(f_{\mu\bar{\nu}}) \, dV$ where dV is the volume form of the torus. Using Eq. (22) this means that $\det(f) = F_2$. In terms of h, the form K is given by:

$$K = * \frac{(\kappa^2)}{2} = \frac{i}{2} \left[f_{\mu_1\bar{\nu}_1} \, f_{\mu_2\bar{\nu}_2} \right] \epsilon^{\mu_1\mu_2}{}_{\alpha} \, \epsilon^{\bar{\nu}_1\bar{\nu}_2}{}_{\bar{\beta}} \, dz^\alpha \wedge d\bar{z}^\beta \quad (129)$$

After some manipulation of the ϵ tensors and taking into account the symmetries of the metric, we find that:

$$\frac{K \wedge K \wedge K}{3!} = dV \; \det(f)^2 = dV \, F_2^2 \quad (130)$$

Using this is Eq. (128) gives the result:

$$\sqrt{\det g_{int}} = \frac{1}{\sqrt{F_2^3 F_6^3}} F_2^2 = \sqrt{\frac{F_2}{F_6^3}} \quad (131)$$

B The equations of motion

The purpose of this appendix is to show explicitly that the classical configurations considered in this paper do indeed satisfy the equations of motion. This computation is straightforward in principle but, unless properly organized, it can be prohibitively tedious. We restrict the attention to the case of D2-branes at angles, because the inclusion of D6-branes involves no essential new features.

In Sec. 4 we showed that the classical configurations Eq. 71–74 are supersymmetric in spacetime. These supersymmetric configurations are completely characterized by the Kähler metric $g_{\mu\bar{\nu}}$ of the compact space. It is important to note that the dependence of $g_{\mu\bar{\nu}}$ on the uncompactified coordinates is not restricted by supersymmetry. However, our *ansatz* Eq. 17-23 is more restrictive: the $g_{\mu\bar{\nu}}$ are given in terms of the X_i that in turn are *harmonic functions* of the external coordinates. We can use Eq. 45 to express this additional requirement conveniently as:

$$\partial^2(F^{\frac{1}{2}} g^{\mu\bar{\nu}}) = 0 \tag{132}$$

It is easy to use the Bianchi identities to show that this harmonic function property is indeed a necessary condition to satisfy the equation of motion. In this section we will verify that this condition is also sufficient.

Note that in Eq. 132, and repeatedly in the following, the ∂_i denote derivatives with respect to the *Cartesian coordinates* in the noncompact space, and the index contraction implied in the notation $\partial^2 \equiv \partial_i\partial^i$ is with respect to the Cartesian flat metric in three dimensions, *i.e.* δ_i^j.

B.1 Preliminaries

We use the notation explained in the beginning of Sec.4. Our strategy is to use the *ansatz* Eq. 71–74 to express all quantities in terms of the functions $g_{\mu\bar{\nu}}$ and F. Subsequently the harmonic function property Eq. 132 is employed to simplify the expressions.

Equations of motion: We write the part of the Lagrangian that is needed for the present purposes as:

$$\mathcal{L} = \sqrt{-g}[e^{-2\Phi}(R + 4(\nabla\Phi)^2) - \frac{1}{48}F_4^2] \tag{133}$$

where F_4 is the 4-form field strength that couples to the D2-branes. The corresponding equations of motion can be written:

$$\begin{aligned} R &= -4\nabla^2\Phi + 4(\nabla\Phi)^2 && (134)\\ R_{IJ} &= -2\nabla_I\nabla_J\Phi + \frac{1}{12}e^{2\Phi}(F_I{}^{KLM}F_{JKLM} - \frac{1}{8}g_{IJ}F^{KLMN}F_{KLMN}) && (135)\\ 0 &= \frac{1}{\sqrt{-g}}\partial_I(\sqrt{-g}F^{IJKL}) && (136) \end{aligned}$$

We first consider the left hand side of these equations.

Curvature: The nonvanishing components of the Christoffel symbols are:

$$\Gamma_{ijk} = \frac{1}{4}F^{-\frac{1}{2}}(\delta_{ik}\partial_j F + \delta_{ij}\partial_k F - \delta_{jk}\partial_i F) \quad (137)$$

$$\Gamma_{i00} = -\frac{1}{4}F^{-\frac{3}{2}}\partial_i F = -\Gamma_{0i0} \quad (138)$$

$$\Gamma_{i\mu\bar{\nu}} = -\frac{1}{2}\partial_i g_{\mu\bar{\nu}} = -\Gamma_{\mu i\bar{\nu}} = -\Gamma_{\bar{\nu}i\mu} \quad (139)$$

We then use:

$$R_{IJ} = \partial_K \Gamma^K_{IJ} - \partial_I \Gamma^K_{KJ} + \Gamma^K_{IJ}\Gamma^L_{KL} - \Gamma^K_{LJ}\Gamma^L_{KI} \quad (140)$$

to find the nonvanishing components of the Ricci tensor:

$$R_{00} = -\frac{1}{4}F^{-2}\partial^2 F + \frac{1}{8}F^{-3}(\partial F)^2 \quad (141)$$

$$\begin{aligned} R_{ij} &= -\frac{1}{2}F^{-1}\partial_i\partial_j F - \frac{1}{4}\delta_{ij}F^{-1}\partial^2 F + \frac{5}{8}F^{-2}\partial_i F\partial_j F + \frac{1}{8}\delta_{ij}F^{-2}(\partial F)^2 + \\ &+ \frac{1}{2}\partial_i g^{\mu\bar{\nu}}\partial_j g_{\mu\bar{\nu}} \end{aligned} \quad (142)$$

$$R_{\mu\bar{\nu}} = -\frac{1}{2}F^{-\frac{1}{2}}\partial^2 g_{\mu\bar{\nu}} - \frac{1}{4}F^{-\frac{3}{2}}\partial F\partial g_{\mu\bar{\nu}} + \frac{1}{2}F^{-\frac{1}{2}}\partial g_{\mu\bar{\rho}}g^{\bar{\rho}\lambda}\partial g_{\lambda\bar{\nu}} \quad (143)$$

In the evaluation we used:

$$\Gamma^I_{Ik} = \frac{1}{2g}\partial_k g = F^{-1}\partial_k F \quad (144)$$

The harmonic function property Eq. 132 can be employed to show:

$$\partial^2 g_{\mu\bar{\nu}} = [\frac{1}{2}F^{-1}\partial^2 F - \frac{1}{4}F^{-2}(\partial F)^2]g_{\mu\bar{\nu}} - F^{-1}\partial F\partial g_{\mu\bar{\nu}} + 2\partial g_{\mu\bar{\rho}}g^{\bar{\rho}\lambda}\partial g_{\lambda\bar{\nu}} \quad (145)$$

$$\partial g^{\mu\bar{\nu}}\partial g_{\mu\bar{\nu}} = F^{-1}\partial^2 F - \frac{3}{4}F^{-2}(\partial F)^2 \quad (146)$$

The first relation allows us to write the Ricci tensor with indices in the compact directions as:

$$R_{\mu\bar{\nu}} = [-\frac{1}{4}F^{-\frac{3}{2}}\partial^2 F + \frac{1}{8}F^{-\frac{5}{2}}(\partial F)^2]g_{\mu\bar{\nu}} + \frac{1}{4}F^{-\frac{3}{2}}\partial F\partial g_{\mu\bar{\nu}} - \frac{1}{2}F^{-\frac{1}{2}}\partial g_{\mu\bar{\rho}}g^{\bar{\rho}\lambda}\partial g_{\lambda\bar{\nu}} \quad (147)$$

and the second relation is needed to find the Ricci scalar:

$$R = g^{00}R_{00} + g^{ij}R_{ij} + 2g^{\mu\bar{\nu}}R_{\mu\bar{\nu}} \quad (148)$$

$$= -F^{-\frac{3}{2}}\partial^2 F + \frac{3}{4}F^{-\frac{5}{2}}(\partial F)^2 \quad (149)$$

B.2 The equations of motion

At this point we have evaluated the left hand side of the equations of motion Eq. 134-136, *i.e.* the gravity contribution. We now proceed to consider the right hand side, *i.e.* the matter contribution.

The gauge field equation The simplest and most instructive of the equations of motion is Eq. 136 for the 4-form field strength. The only component that is nontrivial is:

$$\frac{1}{\sqrt{-g}}\partial_I(\sqrt{-g}F^{It\mu\bar{\nu}}) = F^{-1}\partial_i(g^{ij}Fg^{\bar{\nu}\nu}g^{\bar{\mu}\mu}\partial_j F^{-\frac{1}{2}}g_{\bar{\mu}\nu}) = -F^{-1}\partial^2 F^{\frac{1}{2}}g^{\mu\bar{\nu}} = 0 \qquad (150)$$

as it should be. Note that the final step requires exactly the harmonic function property Eq. 132. This calculation therefore demonstrates the necessity of this condition. In fact, it is precisely the Bianchi identity that is verified in this step.

The dilaton equation: The *ansatz* Eq. 17-23 gives:

$$\nabla^2\Phi = \frac{1}{4}F^{-\frac{3}{2}}\partial^2 F - \frac{1}{8}F^{-\frac{5}{2}}(\partial F)^2 \qquad (151)$$

$$(\nabla\Phi)^2 = \frac{1}{16}F^{-\frac{5}{2}}(\partial F)^2 \qquad (152)$$

These expressions, and Eq. 149 for the Ricci scalar, indeed satisfy the dilaton equation Eq. 134.

Temporal part of the Einstein equation: A short calculation that uses Eq. 87 and Eq. 146 gives:

$$\frac{1}{24}F_4^2 \equiv \frac{1}{24}F^{KLMN}F_{KLMN} = F^{-2}\partial^2 F - F^{-3}(\partial F)^2 \qquad (153)$$

and therefore:

$$\frac{1}{12}e^{2\Phi}(F_0{}^{KLM}F_{0KLM} - \frac{1}{8}g_{00}F^{KLMN}F_{KLMN}) = \frac{1}{48}e^{2\Phi}g_{00}F_4^2 = -\frac{1}{4}[F^{-2}\partial^2 F - F^{-3}(\partial F)^2] \qquad (154)$$

We must also calculate the covariant derivatives of the dilaton:

$$-2\nabla_0\nabla_0\Phi = 2\Gamma^k_{00}\partial_k\Phi = -\frac{1}{8}F^{-3}(\partial F)^2 \qquad (155)$$

These expressions, and the R_{00} from Eq. 141, satisfy the temporal part of the gravition equation Eq. 135.

The external part of the Einstein equation: The stress tensor of the 4-form field strength is:

$$\frac{1}{12}e^{2\Phi}(F_i{}^{KLM}F_{jKLM} - \frac{1}{8}g_{ij}F^{KLMN}F_{KLMN}) \qquad (156)$$

$$= -\frac{1}{4}\delta_{ij}[F^{-1}\partial^2 F - F^{-2}(\partial F)^2] - \frac{1}{8}F^{-2}\partial_i F\partial_j F + \frac{1}{2}\partial_i g_{\mu\bar{\nu}}\partial_j g^{\mu\bar{\nu}} \qquad (157)$$

and we also need:

$$-2\nabla_i\nabla_j\Phi = -2(\partial_i\partial_j - \Gamma^k_{ji}\partial_k)\Phi = -\frac{1}{2}F^{-1}\partial_i\partial_j F + \frac{3}{4}F^{-2}\partial_i F\partial_j F - \frac{1}{8}\delta_{ij}F^{-2}(\partial F)^2 \qquad (158)$$

The external part of the Einstein equation Eq. 135 can now be verified by adding these equations, using Eq. 146, and comparing with the R_{ij} given in Eq. 142.

The internal part of the Einstein equations: We finally consider the Einstein equation with indices in the compact direction. The energy momentum carried by the 4-form field strength is:

$$\frac{1}{12}e^{2\Phi}(F_{\mu}{}^{KLM}F_{\bar{\nu}KLM} - \frac{1}{8}g_{\mu\bar{\nu}}F^{KLMN}F_{KLMN}) \tag{159}$$

$$= [\frac{1}{8}F^{-\frac{5}{2}}(\partial F)^2 - \frac{1}{4}F^{-\frac{3}{2}}\partial^2 F]g_{\mu\bar{\nu}} + \frac{1}{2}F^{-\frac{3}{2}}\partial F\partial g_{\mu\bar{\nu}} - \frac{1}{2}F^{-\frac{1}{2}}\partial g_{\mu\bar{\rho}}g^{\bar{\rho}\lambda}\partial g_{\lambda\bar{\nu}} \tag{160}$$

and we also need:

$$-2\nabla_\mu\nabla_{\bar{\nu}}\Phi = 2\Gamma^k_{\mu\bar{\nu}}\partial_k\Phi = -\frac{1}{4}F^{-\frac{1}{2}}\partial g_{\mu\bar{\nu}} \tag{161}$$

Adding these two terms and comparing with $R_{\mu\bar{\nu}}$ in Eq. 147, we verify the internal part of the Einstein equation Eq. 135.

This completes the verification of the equations of motion.

References

[1] G. Papadopoulos and P. K. Townsend, Intersecting M-Branes. *Phys. Lett. B*, 379: 273, 1996. hep-th/9603087.

[2] A. Strominger and C. Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys. Lett. B*, 379:99–104, 1996. hep-th/9601029.

[3] F. Larsen and F. Wilczek. Internal structure of black holes. *Phys. Lett. B*, 375:37–42, 1996. hep-th/9511064.

[4] C. Callan and J.Maldacena. D-brane approach to black hole quantum mechanics. *Nucl. Phys. B*, 472:591–610, 1996. hep-th/9602043.

[5] G.T. Horowitz and A. Strominger. Counting states of near-extremal black holes. *Phys. Rev. Lett*, 77:2369–2371, 1996. hep-th/9602051.

[6] R. Dijkgraaf, E. Verlinde, and H. Verlinde. BPS spectrum of the five-brane and black hole entropy. *Nucl. Phys. B*, 486:77–88, 1996. hep-th/9603126.

[7] M. Berkooz, M.R. Douglas, and R.G. Leigh. Branes intersecting at angles. *Nucl. Phys.B*, 480:265–278, 1996. hep-th/9606139.

[8] V. Balasubramanian and R.G. Leigh. D-branes, moduli and supersymmetry. *Phys. Rev. D*, 55:6415–6422, 1997. hep-th/9611165.

[9] J. P. Gauntlett, G. W. Gibbons, G. Papadopoulos, and P. K. Townsend. Hyperkähler manifolds and multiply intersecting branes. hep-th/9702202.

[10] K. Behrndt and M. Cvetic. BPS saturated bound states of tilted p-branes in type II string theory. hep-th/9702205.

[11] J.C. Breckenridge, G. Michaud, and R.C. Myers. New angles on D-branes. hep-th/9703041.

[12] M. Costa and M. Cvetic. Nonthreshold D-brane bound states and black holes with nonzero entropy. hep-th/9703204.

[13] N. Hambli. Comments on Dirichlet branes at angles. hep-th/9703179.

[14] A.A. Tseytlin. Harmonic superposition of M-branes. *Nucl. Phys.B*, 475:149–163, 1996. hep-th/9604035.

[15] V. Balasubramanian, F. Larsen, and R.G. Leigh. Work in progress.

[16] J. Polchinski. TASI lectures on D-branes. hep-th/9611050.

[17] V. Balasubramanian and F. Larsen. On D-branes and black holes in four dimensions. *Nucl. Phys.B*, 478:199, 1996. hep-th/9604189.

[18] K. Behrndt, G.L. Cardoso, B. de Wit, R. Kallosh, D. Lüst, and T. Mohaupt. Classical and quantum N=2 supersymmetric black holes. *Nucl. Phys. B*, 488:236–260, 1997. hep-th/9610105.

[19] A. Giveon, M. Porrati, and E. Rabinovici. Target space duality in string theory. *Phys. Rept.*, 244: 77-202, 1994. hep-th/9401139.

[20] M. Cvetič and A. Tseytlin. Solitonic strings and BPS saturated dyonic black holes. *Phys. Rev. D*, 53:5619–5633, 1996. hep-th/9512031.

[21] R.R. Khuri M.J. Duff and J.X. Lu. String solitons. *Phys. Rep.*, 259:213–326, 1995. hep-th/9412184.

[22] E. Cremmer and B. Julia. The SO(8) supergravity. *Nucl. Phys.B*,159:141, 1979.

[23] C. M. Hull and P. K. Townsend. Unity of superstring dualities. *Nucl. Phys. B*, 438:109, 1995. hep-th/9410167.

[24] C. Hull and M. Cvetič. Black holes and U-duality. *Nucl. Phys. B*, 480:296–316, 1996. hep-th/9606193.

[25] S.Ferrara and R. Kallosh. Universality of supersymmetry attractors. *Phys. Rev. D*, 54:1525–1534, 1996. hep-th/9603090.

[26] R. Kallosh and B. Kol. E(7) symmetric area of the black hole horizon. *Phys. Rev. D*, 53:5344–5348, 1996. hep-th/9602014.

[27] V. Balasubramanian and F. Larsen. Relativistic brane scattering. hep-th/9703039, to appear in Nucl. Phys. B.

[28] I. Klebanov and A. Tseytlin. Intersecting M-branes as four-dimensional black holes. *Nucl. Phys.B*, 475:179–192, 1996. hep-th/9604166.

[29] J. Maldacena. N=2 extremal black holes and intersecting branes. hep-th/9611163.

[30] K. Behrndt and T. Mohaupt. Entropy of N=2 black holes and their M-brane description. hep-th/9611140.

[31] M.B. Green, J.H. Schwarz, and E. Witten. *Superstring Theory*. Cambridge University Press, 1987.

[32] S. Das and S. Mathur. Excitations of D-strings, entropy and duality. hep-th/9601152.

[33] J. Maldacena and L. Susskind. D-branes and fat black holes. hep-th/9604042.

[34] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde. Elliptic genera of symmetric products and second quantized strings. hep-th/9608096.

[35] R. Dijkgraaf, E. Verlinde, and H. Verlinde. Counting dyons in N=4 string theory. hep-th/9607026.